Algebraic Geometry

Professor: Anand Sawant
Notes By: Ajay Prajapati

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§1. Preliminaries

Here we give the proof of important facts which are used in the proof of results of next section.

Theorem 1.1. [Ex II.2.18(c)] Let X=Spec A, Y=Spec B, $\varphi : A \longrightarrow B$ be a ring homomorphism and $f : Y \longrightarrow X$ be the induced morphism of schemes. If φ is surjective then f is a closed immersion.

Proof. f is continuous because $f^{-1}(D(a)) = D(\varphi(a))$. f is injective because prime ideals of B are in one to one correspondence with the prime ideals of A containing $ker(\varphi)$. It is closed since $f(V(I)) = V(\varphi^{-1}(I))$. So it is homeomorphism onto its image. Now we want to check $f^\# : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is surjective. This is equivalent to checking if the map of local rings is surjective. Let $\mathfrak{p} \in$ Spec A. Then $f^\#_\mathfrak{p} : A_\mathfrak{p} \longrightarrow (f_*\mathcal{O}_Y)_\mathfrak{p}$. Now (last equality is as $A_\mathfrak{p}$ modules)

$$(f_*\mathcal{O}_Y)_\mathfrak{p} = \varinjlim_{V \ni \mathfrak{p}} \mathcal{O}(f^{-1}(V)) = \varinjlim_{a \notin \mathfrak{p}} \mathcal{O}(D(\varphi(a))) = \varinjlim_{a \notin \mathfrak{p}} B_{\varphi(a)} = B \otimes_A A_\mathfrak{p}$$

Since $\otimes$ is right exact, the map $A_\mathfrak{p} \longrightarrow B \otimes_A A_\mathfrak{p}$ is surjective. □

Definition 1.2. Let A and B be two local rings. A is said to *dominate* B if $A \subseteq B$ as subring and $\mathfrak{m}_A \subseteq \mathfrak{m}_B$. i.e. $\mathfrak{m}_A = A \cap \mathfrak{m}_B$.

Theorem 1.3. [AM, Ex 5.27] Let K be a field and let Σ be the set of all local subrings of K. If Σ is ordered by relation of domination, then Σ has maximal elements and $A \in \Sigma$ is maximal iff A is a valuation ring.

Proof. Since $K \in \Sigma$, it is non-empty. Let $A_1 \leq A_2 \leq \ldots$ be a chain and let $A = \cup_i A_i$ and $\mathfrak{m} = \cup_i \mathfrak{m}_i$. Let $f \in A \backslash \mathfrak{m}$. Then $f \in A_i \backslash \mathfrak{m}_i$. Hence $f^{-1} \in A_i \subseteq A$. So A is a local ring with the maximal ideal $\mathfrak{m}$. By Zorn's lemma, there exists maximal elements of Σ.

Let $A, \mathfrak{m} \in \Sigma$ is a maximal element and let $f \in K \backslash A$. Then $A \subseteq A_f$ as subring. Clearly, $\mathfrak{m}A_f$ is proper ideal of A_f (otherwise $\mathfrak{m}A_f = A_f \implies f \in \mathfrak{m}$), then it is contained in the maximal ideal $\mathfrak{n}$ of A_f. Since A is maximal in Σ, $A = A_f$. This implies $f^{-1} \in A$.

Suppose A is a valuation ring of K. Then A is a local ring. Suppose B is another local ring dominating it. If $f \in B \backslash A$ then $f^{-1} \in A$ is a non-unit. So $f^{-1} \in \mathfrak{m}_A \subseteq \mathfrak{m}_B$ is non-unit in B. This cannot happen since $f, f^{-1} \in B$. Hence $B = A$. □

Lemma 1.4. [Ex II.3.11(a)] A closed immersion is stable under base extension.

Proof. Let $f : X \longrightarrow Y$ be a closed immersion and $f' : X' \longrightarrow Y'$ be the base extension to $g : Y' \longrightarrow Y$. As topological space, $X' = g^{-1}(X)$ is closed as $X \subseteq Y$ is closed. We want to check surjectivity of stalks $f_P^{\#} : \mathcal{O}_{Y',f'(P)} \longrightarrow f'_*\mathcal{O}_{X',P}$ for all $P \in X'$. Let $P \in X'$ and choose an open affine $U = \operatorname{Spec} B \subseteq Y'$ containing $f'(P)$ such that $g(U)$ is small enough to be contained in an open affine $V = \operatorname{Spec} A \subseteq Y$. Now $f^{-1}(V) = \operatorname{Spec} A/I$ for some ideal I of A. Now $\operatorname{Spec}(B \otimes_A A/I) \subseteq X'$ is a neighbourhood of P. Since $B \otimes_A A/I \cong B/IB$ and map $B \longrightarrow B/IB$ is surjective, the map of stalks is surjective as well at P by 1.1. □

Lemma 1.5. [Ex II.3.11(c)] Let Y be a closed subset of a scheme X and give Y the reduced induced structure. If Y' is another closed subscheme such that $sp(Y) = sp(Y')$ then closed immersion $Y \longrightarrow X$ factors through Y.

Proof. Suppose $X = \operatorname{Spec} A$ is affine. Then $Y = \operatorname{Spec} A/I$ for the ideal $I = \cap_{\mathfrak{p} \in Y}\mathfrak{p}$ since Y has reduced induced structure. Also let $Y' = \operatorname{Spec} A/J$. Since $sp(Y) = sp(Y')$, we have $\sqrt{J} = I$. Now the natural map $A \longrightarrow A/I$ factors as $A \longrightarrow A/J \longrightarrow A/I$. Hence statement of lemma is true for X. If X is not affine, cover X by affine opens and then glue which will be compatible with the closed immersions $Y \cap \operatorname{Spec} A_i \longrightarrow Y' \cap \operatorname{Spec} A_i \longrightarrow X \cap \operatorname{Spec} A_i$. □

Theorem 1.6. [Ex II.3.1] A morphism $f : X \longrightarrow Y$ is locally of finite type $\iff$ for *every* affine open $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ is covered by open affine subsets $U_j = \operatorname{Spec} A_j$, where each A_j is finitely generated B-algebra.

Proof. ($\implies$) Let $Y = \cup_i V_i$, $V_i = \operatorname{Spec} B_i$ such that for each i, $f^{-1}(V_i) = \cup_j V_{ij}$, $V_{ij} = \operatorname{Spec} A_{ij}$ where A_{ij} is finitely generated B_i-algebra. Let $V = \operatorname{Spec} B \subseteq Y$ is given. Each $V_i \cap V$ can be covered by distinguished open sets $\operatorname{Spec}(B_i)_{b_k}$. Consider b_k as an element of A_{ij} under the homomorphisms $B_i \longrightarrow A_{ij}$, then $f^{-1}(\operatorname{Spec}(B_i)_{b_k}) = \cup_j \operatorname{Spec}(A_{ij})_{b_k}$. Now $(A_{ij})_{b_k}$ is finitely generated $(B_i)_{b_k}$-algebra.

We have covered $V = \operatorname{Spec} B$ by open affines $\operatorname{Spec} C_i$ whose preimages are covered by open affines $\operatorname{Spec} D_{ij}$ such that each D_{ij} is finitely generated C_i-algebra. Now given $\mathfrak{p} \in \operatorname{Spec} B$, it is contained in some $\operatorname{Spec} C_i$. Also since $\operatorname{Spec} C_i$ is open, there exists a distinguished open set of $\mathfrak{p} \in \operatorname{Spec} B_{g_\mathfrak{p}}$ of $\operatorname{Spec} B$ contained in $\operatorname{Spec} C_i$. Identify $g_\mathfrak{p}$ with its images under the maps $B \longrightarrow C_i$ and $B \longrightarrow D_{ij}$. Then $B_{g_\mathfrak{p}} \cong (C_i)_{g_\mathfrak{p}}$ (By the compatibility condition of a presheaf) and $f^{-1}(\operatorname{Spec} B_{g_\mathfrak{p}}) = \cup_j \operatorname{Spec}(D_{ij})_{g_\mathfrak{p}}$ where $(D_{ij})_{g_\mathfrak{p}}$ is finitely generated $(C_i)_{g_\mathfrak{p}} \cong B_{g_\mathfrak{p}}$-algebra. Hence $(D_{ij})_{g_\mathfrak{p}}$ is f.g. B-algebra by adding the generator $1/g_\mathfrak{p}$.

($\impliedby$) Obvious from definitions. □

Theorem 1.7. [Ex II.3.2] A morphism $f : X \longrightarrow Y$ is quasi-compact $\iff$ for *every* affine open subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.

Proof. ($\Longrightarrow$) Let $Y = \cup_i V_i$, $V_i = \operatorname{Spec} B_i$ such that for each i, $f^{-1}(V_i)$ is quasi-compact. Let $V = \operatorname{Spec} B$ is any open affine of Y. Cover V by distinguished open sets which are also contained in one of V_i's. Since affine schemes are quasi-compact, it is sufficient to prove that preimages of distinguished open sets are quasi-compact. (as $\operatorname{Spec} B$ will be covered by finitely many of them).

Hence we are reduced to the case where X is quasi-compact and $Y = \operatorname{Spec} B$ is affine. Cover X by finite affines $\operatorname{Spec} A_i$ and let f_i be the restriction of f to $\operatorname{Spec} A_i$. Now let $D(g) \subseteq Y$ be distinguished open then $f^{-1}(D(g)) = \cup_i f_i^{-1}(D(g)) = \cup_i D(f_i^{\#}(g))$. Now each $D(f_i^{\#}(g))$ is quasi-compact (since it is affine) hence $f^{-1}(D(g))$ is quasi-compact.

($\Longleftarrow$) Obvious from definitions. □

Theorem 1.8. [Ex II.3.3] A morphism f is of finite type $\iff$ for *every* open affine subset $V = \operatorname{Spec} B \subseteq Y$, $f^{-1}(V) = \cup_i \operatorname{Spec} A_i$ is finite union where each A_i is finitely generated B-algebra.

Proof. ($\Longrightarrow$) This follows directly from 1.6 and 1.7 and the fact that if a space is covered by finitely many affine opens then it is quasi-compact.

($\Longleftarrow$) Obvious from definitions. □

Lemma 1.9.
1. Ex II.3.13- A closed immersion is a morphism of finite type.
2. A composition of two morphisms of finite type is of finite type.
3. Morphisms of finite type are stable under base extension.

Proof. 1) Let $f : X \longrightarrow Y$ be a closed immersion. Cover Y by affine opens V_i =Spec A_i. Then $f^{-1}(V_i)$ =Spec A_i/I for some ideal $I \subseteq A_i$. Clearly A_i/I is finitely generated A_i-algebra.

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2) Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms of finite type, let $h = g \circ f$ and $V = \operatorname{Spec} C \subseteq Z$. Then by theorem 1.8, $g^{-1}(V) = \cup_i \operatorname{Spec} B_i$ s.t. B_i is f.g. C-algebra. Now $f^{-1}(B_i) = \cup_j \operatorname{Spec} A_{ij}$ where A_{ij} is f.g. B_i-algebra. Hence $h^{-1}(V) = \cup_{i,j} \operatorname{Spec} A_{ij}$ where A_{ij} is finitely generated C-algebra.

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3) Let $f : X \longrightarrow Y$ be a morphism and $f' : X' \longrightarrow Y'$ be the base extension

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Let $U = \operatorname{Spec} B$ be affine open in Y s.t. $g^{-1}(U) \neq \emptyset$ and let $V = \operatorname{Spec} A' \subseteq g^{-1}(U)$. Let $f^{-1}(U) = \cup_i \operatorname{Spec} A_i$ is finite union where A_i is finitely generated B-algebra. Now $f'^{-1}(V) = \cup_i(\operatorname{Spec} A' \otimes_B A_i)$. If $\{b_1, \ldots b_r\}$ is finite generating set of A_i as an B-algebra then $\{b_1 \otimes 1, \ldots b_r \otimes 1\}$ is finite generating set $(\operatorname{Spec} A' \otimes_B A_i)$ as an A'-algebra.

Now cover Y with open affines $\{U_i\}_i$. Then we can cover $g^{-1}(U_i)$ by open affines $V_{ij} = \operatorname{Spec} B'_{ij}$. And $f'^{-1}(V_{ij})$ can be covered by finitely many open affines $\operatorname{Spec} A'_{ijk}$ such that A'_{ijk} is finitely generated B'_{ij}-algebra. Hence f' is of finite type. □

Now we study maps from the specturm of valuation rings to a scheme. This will be used in giving valuative criterion of properness and seperatedness.

Lemma 1.10. Let K be a field and X a scheme. Then to give a morphism Spec $K \longrightarrow X$ is same as giving a point $x_1 \in X$ and inclusion of fields $k(x_1) \subseteq K$.

Proof. ($\Longrightarrow$) Suppose a morphism from Spec $K \longrightarrow X$ given and x_1 is the image. Then we have local homomorphism of local rings $\mathcal{O}_{x_1,X} \longrightarrow K$ which induces inclusion $k(x_1) \subseteq K$.

($\Longleftarrow$) Suppose $k(x_1) \subseteq K$ is given. This induces local homomorphism $\mathcal{O}_{x_1,X} \longrightarrow K$ which induces morphism Spec $K \longrightarrow X$ taking t_1 to x_1. □

Lemma 1.11. Let R be a valuation ring with quotient field K, let X be a scheme. To give a morphism Spec $R \longrightarrow X$ is same as giving two points $x_0, x_1 \in X$ with $x_0 \in \overline{\{x_1\}}$ (x_0 is specialization of x_1), and an inclusion of fields $k(x_1) \subseteq K$ such that R dominates the local ring $\mathcal{O}_{x_0,Z}$ of x_0 on the subscheme $Z = \overline{\{x_1\}}$ of X with reduced induced structure.

Proof. ($\Longrightarrow$) Suppose Spec $R \longrightarrow X$ is given. Let $t_0 = m_R$ and $t_1 = (0)$ is the generic point. Since Spec R is reduced, by lemma 1.5 morphism Spec $R \longrightarrow X$ factors through Z. Now Z is reduced and irreducible hence integral with the function field $k(x_1) = \mathcal{O}_{x_1,Z}$. We have local homomorphism of local rings $\mathcal{O}_{x_0,Z} \longrightarrow R$ compatible with the inclusion $k(x_1) \subseteq K$. Thus $\mathcal{O}_{x_0,Z} \longrightarrow R$ is injective and R dominates $\mathcal{O}_{x_0,Z}$.

$$\begin{array}{ccc} \mathcal{O}_{x_0,Z} & \longrightarrow & R \\ \big\downarrow & & \big\downarrow \\ k(x_1) & \hookrightarrow & K \end{array}$$

($\Longleftarrow$) Given data consisting of x_0, x_1, the inclusion $k(x_1) \subseteq K$ and that R dominates $\mathcal{O}$, we have inclusion $\mathcal{O} \longrightarrow R$ which induces $\operatorname{Spec} R \longrightarrow \operatorname{Spec} \mathcal{O}$.

Choose an affine neighbourhood $U = \operatorname{Spec} A$ of x_0 in Z, we have $\mathcal{O} = A_{\mathfrak{p}}$ where $x_0 = \mathfrak{p}$. Thus we have natural map $A \longrightarrow \mathcal{O}$ which gives $\operatorname{Spec} \mathcal{O} \longrightarrow \operatorname{Spec} A$. Compose this with natural maps $\operatorname{Spec} A \longrightarrow Z \longrightarrow X$. □

Lemma 1.12. Let $f : X \longrightarrow Y$ be quasi-compact morphism of schemes. Then the subset $f(X)$ is closed $\iff$ $f(X)$ is stable under specialization.

Proof. ($\implies$) Obvious from definitions.

($\impliedby$) The map $X_{red} \longrightarrow X \longrightarrow Y$ is still quasi-compact. By lemma 1.5, this map factors as $X_{red} \longrightarrow \overline{f(X)} \longrightarrow Y$ where $\overline{f(X)}$ is given reduced induced structure and the first map is still quasi-compact. So WLOG, we can assume that X and Y are reduced and $\overline{f(X)} = Y$. Let $y \in Y$ is a point. We want to show that $y \in f(X)$. Let U be an affine neighbourhood of y, then by 1.7, $f^{-1}(U)$ is quasi-compact. So further we can assume Y is affine.

Cover X by finitely many open affines X_i. Since $y \in \overline{f(X)}$, we have $y \in \overline{f_i(X)}$ for some i. Let $X_i = \operatorname{Spec} A$. Let $Y_i = \overline{f_i(X)}$ with the reduced induced structure. Then Y_i is also affine. Let $X_i = \operatorname{Spec} A$ and $Y_i = \operatorname{Spec} B$, then the ring homomorphism $\varphi : B \longrightarrow A$ which induces f is injective (Let $\varphi(b) = 0 \implies f^{-1}(D(b)) = D(\varphi(b)) = D(0) = \varnothing \implies D(b) = \varnothing$ since $X_i \longrightarrow Y_i$ is dominant, so $b \in \mathfrak{N}(B) \implies b = 0$ since B is reduced). The point y corresponds to a prime ideal $\mathfrak{p}$ of B and let $\mathfrak{p}' \subseteq \mathfrak{p}$ be the minimal prime ideal. Then $\mathfrak{p}'$ corresponds to a point $y' \in Y_i$ which specializes to y.

Claim: $y' \in f(X_i)$.

Since localization is an exact functor, we have $B_{\mathfrak{p}'} \subseteq (A)_{\mathfrak{p}'} \cong A \otimes_B B_{\mathfrak{p}'}$. Now $B_{\mathfrak{p}'}$ is a field because $\mathfrak{p}'$ is a minimal prime ideal so $B_{\mathfrak{p}'}$ has only one prime ideal which equals to $\mathfrak{N}(B_{\mathfrak{p}'})$. Now B is reduced so $\mathfrak{N}(B) = 0$. Since localization commutes with taking radicals, we have $\mathfrak{N}(B_{\mathfrak{p}'}) = 0$.

$$\begin{array}{ccc} B & \hookrightarrow & A \\ \downarrow & & \downarrow \\ B_{\mathfrak{p}'} & \hookrightarrow & A \otimes_B B_{\mathfrak{p}'} \end{array}$$

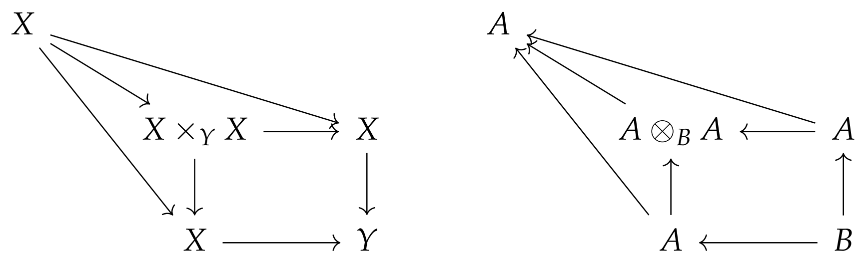
Let $\mathfrak{n}$ be any prime ideal of $A \otimes_B B_{\mathfrak{p}'}$. And let $\mathfrak{n}'$ is its contraction to A. Then by commutativity of above diagram, $\mathfrak{n}' \cap A = \mathfrak{p}'$. In other words, $f(x') = y'$ where $x' = \mathfrak{n}'$. Now the lemma follows from claim. □

§2. Main theorem of elimination theory

Here we give the proof of the **Main theorem of elimination theory** (2.7). The theorem follows easily from 2.6 which is very difficult to prove and most part of this note is devoted in proving this. 2.6 follows from **Valuative criterion of properness** (2.5) and the properties of morphisms of finite type (lemma 1.9).

Theorem 2.1. If $f : X \longrightarrow Y$ is any morphism of affine schemes, then f is seperated.

Proof. Let X=Spec A, Y=Spec B. Then we have following diagram:



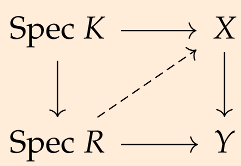
The map $A \times_B A \longrightarrow A, a \otimes a' \mapsto aa'$ is surjective. Hence by 1.1, $\Delta : X \longrightarrow X \times_Y X$ is a closed immersion. □

Corollary 2.2. A morphism $f : X \longrightarrow Y$ of arbitrary schemes is seperated $\iff$ the image of Δ is closed in $X \times_Y X$.

Proof. ($\Longrightarrow$) Obvious from definition.

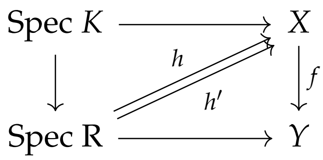
($\Longleftarrow$) In order to prove that Δ is closed immersion, it is sufficient to check that it is homeomorphism onto its image and map of sheaves is surjective. Let $p_1 : X \times_Y X \longrightarrow X$ be the first projection. Then $p_1 \circ \Delta = Id_X$ which shows Δ is homeomorphism onto its image. Note that surjectivity of sheaves is a local property and locally a map of schemes is given by a map between affine schemes. By theorem 2.1, locally the maps of sheaves is surjective. (exact argument is similar to the proof of 1.4). □

Theorem 2.3 (Valuative criterion of separatedness). Let $f : X \longrightarrow Y$ be a morphism of schemes. Then f is seperated iff the following holds: $\Delta : X \longrightarrow X \times_Y X$ is quasi-compact and for any valuation ring R with quotient field K, in the following commutative diagram



there is atmost one lifting Spec $R \longrightarrow X$ making the whole diagram commute.

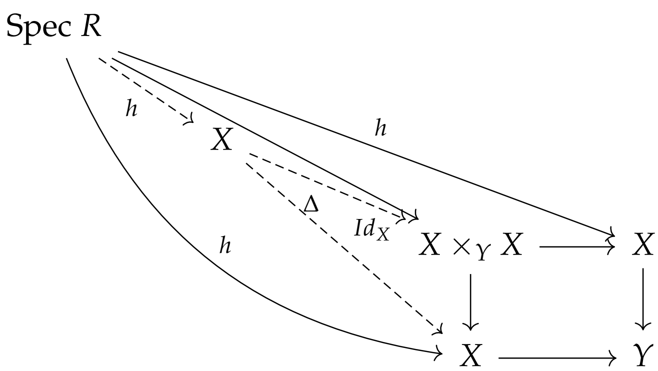
Proof. ($\Longrightarrow$) Suppose there are two morphisms h, h' making following digaram commute



By universal property of fibered products, we obtain h'' :Spec $R \longrightarrow X \times_Y X$. The generic point t_1 of Spec R has image in diagonal (since restriction of h, h' to Spec K is same). Since $\Delta(X)$ is closed, $h''(t_0) \in \Delta(X)$. So both h, h' send t_0, t_1 to the same points x_0, x_1 of X. $k(x_1) \subseteq K$ induced by h, h' are also same. Hence $h = h'$ by 1.11.

($\Longleftarrow$) By 2.2, it is sufficient to prove that $\Delta(X)$ is closed. By lemma 1.12, it is sufficient to prove that $\Delta(X)$ is closed under specilization (quasi-compact is given in hypothesis). So let $\xi_1 \in \Delta(X)$ and $\xi_1 \rightsquigarrow \xi_0$ be a specilization.

Let $Z = \overline{\{\xi_1\}}$ with the reduced induced structure. Then Z is reduced and irreducible hence integral with ξ_1 as its generic point. Hence $k(\xi_1) = \mathcal{O}_{\xi_1,Z} = K$ is function field of Z and $\mathcal{O}_{\xi_0,Z} \subseteq K$ as subring (since quotient field of $\mathcal{O}_{\xi_0,Z}$ is K). By theorem 1.3, there is a valuation ring R of K dominating $\mathcal{O}_{\xi_0,Z}$. By lemma 1.11, we obtain a morphism of Spec R into $X \times_Y X$ sending t_0, t_1 to ξ_0, ξ_1. Composing with projections p_1, p_2 gives two morphism of Spec R to X which give same morphism to Y. Also their restriction to Spec K is same since $\xi_1 \in \Delta(X)$.



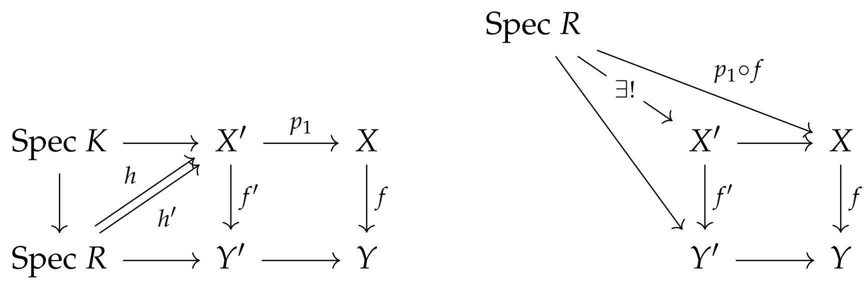
By condition of the theorem, these two morphisms from Spec R to X are same, say h. Now by universal property of fibered products, the map Spec $R \longrightarrow X \times_Y X$ factors as shown above. Hence $\xi_0 \in \Delta(X)$. □

Corollary 2.4. Assume all the schemes below are noetherian.

1. Seperatedness is preserved by base change.

2. Seperatedness is local on the base.i.e. A morphism $f : X \longrightarrow Y$ is seperated iff Y is covered by open subsets V_i s.t. $f^{-1}(V_i) \longrightarrow V_i$ is seperated for each i.
3. Closed immersions are seperated.

Proof. 1. Let $f : X \longrightarrow Y$ be a seperated morphism and $Y' \longrightarrow Y$ be any morphism. We must show h and h' as shown in figure are the same maps.



By 2.3, $p_1 \circ h = p_1 \circ h$. By universal property of fibered products, $h = h'$.

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2. ($\implies$) By first part, $f^{-1}(V_i) \longrightarrow V_i$ is seperated since it is base change by inclusion $V_i \hookrightarrow X$.

($\impliedby$) To check $\Delta(X) \hookrightarrow X \times_Y X$ a closed subset, it suffices to check on an open cover. If $g : X \times_Y X \longrightarrow Y$ is the natural morphism, then open cover V_i of Y induces an open cover $f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$ of $X \times_Y X$. Now $f^{-1}(V_i) \longrightarrow V_i$ implies $f^{-1}(V_i) \longrightarrow f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$ is a closed immersion.

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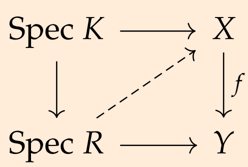
3. Cover $f : X \longrightarrow Y$ be a closed immersion. Cover Y by affine open subsets V_i. Then $f^{-1}(V_i)$ is affine open of X. Now result follows from part (2) and 2.1. □

Theorem 2.5 (Valuative criterion of properness). Let Y be a noetherian scheme, $f : X \longrightarrow Y$ be a morphism of finite type. Then f is proper $\iff$ for any valuation ring R with quotient field K, in the following commutative diagram



there is exactly one lifting Spec $R \longrightarrow X$ making the whole diagram commute.

Proof. ($\implies$) Since f is seperated, uniqueness of lifting will from theorem 2.3 once we know its existence. Consider the base extension Spec $R \longrightarrow Y$ and let

$X' = \text{Spec } R \times_Y X$. By universal property, we get map $\text{Spec } K \longrightarrow X'$.

$$\begin{array}{ccccc} \text{Spec } K & \longrightarrow & X' & \longrightarrow & X \\ & & \downarrow f' & & \downarrow f \\ & & \text{Spec } R & \longrightarrow & Y \end{array}$$

Let ξ_1 be image of the point $t_1 \in \text{Spec } K$. Let $Z = \overline{\{\xi_1\}} \subseteq X'$ be closed subscheme with the reduced induced structure. Since f' is closed, $f'(Z)$ is closed. Since ξ_1 maps to t_0, the generic point of Spec R, we have $f'(Z) = \text{Spec } R$. Let $\xi_0 \in Z$ with $f'(\xi_0) = t_0$. So we get local homomorphism of local rings $R \longrightarrow \mathcal{O}_{\xi_0,Z}$ corresponding to morphism f'.

The function field of Z is $k(\xi_1) \subseteq K$. By theorem 1.3, R is maximal for the dominance relation between local subrings of R. Hence $R \cong \mathcal{O}_{\xi_0,Z}$ and in particular R dominates it ($\mathcal{O}_{\xi_0,Z}$ has field of fractions $k(\xi_1) \subseteq K$ as fields). By lemma 1.11, we get a morphism of $\text{Spec } R \longrightarrow X'$ sending t_0, t_1 to ξ_0, ξ_1. Compose it with $X' \longrightarrow X$ to obtain the required morphism.

($\Longleftarrow$) Suppose condition of this theorem holds. By 2.3, f is seperated and it is finite type by hypothesis. So we want to show it is universally closed. Let $f : Y' \longrightarrow X$ be any map and $f' : X' \longrightarrow Y'$ be base extension. Let Z' be a closed subset of X' and give it reduced induced structure. Want to show $f'(Z')$ is closed in Y'.

$$\begin{array}{ccc} Z' \subseteq X' & \longrightarrow & X \\ \downarrow f' & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

Since $Z' \subseteq X'$ is closed immersion and f' is of finite type, its restriction to Z' is also finite type. In particular $Z' \longrightarrow Y$ is quasi-compact (by definition). Hence by lemma 1.12, we need to show that $f'(Z')$ is stable under specialization.

Let $z_1 \in Z'$, $y_1 = f'(z_1) \in Y'$ and let $y_1 \rightsquigarrow y_0$ be a specialization. Let $Z = \overline{\{y_1\}}$ with reduced induced structure. The quotient field of $\mathcal{O}_{y_1,Z}$ is $k(y_1)$ which is subfield of $K = k(z_1)$. Let R be a valuation ring of K which dominates $\mathcal{O}_{y_1,Z}$ (which exists by theorem 1.3).

By lemma 1.11, we have morphisms making following diagram commute:

$$\begin{array}{ccccc} \text{Spec } K & \longrightarrow & Z & \longrightarrow & X \\ \downarrow & & \downarrow f' & & \downarrow f \\ \text{Spec } R & \longrightarrow & Y' & \longrightarrow & Y \end{array}$$

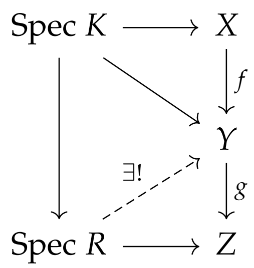
Now see the diagram in the proof of (3) of the next proposition. □

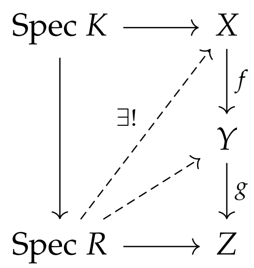
Corollary 2.6. Assume all the schemes below are noetherian.

1. Closed immersions are proper.
2. A composition of proper morphisms is proper.
3. Proper morphisms are stable under base change.

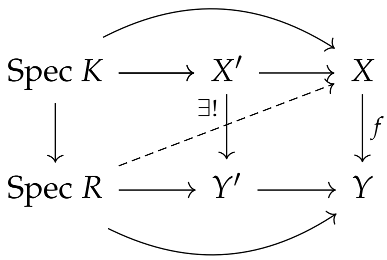
Proof. 1. Let $f : X \longrightarrow Y$ be a closed immersion. By 1.4, f is stable under base extension. In particular, f is universally closed. By 1.9(a), f is of finite type. By 2.4(c), f is seperated.

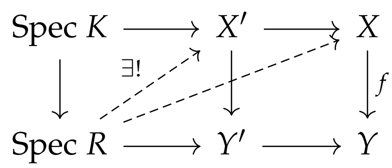
2. Let f and g are proper morphisms. By lemma 1.9(b), $g \circ f$ is of finite type. Hence we can use valuative criterion to check properness of $g \circ f$.





3. Let $f : X \longrightarrow Y$ be a proper morphism and $f' : X' \longrightarrow Y'$ be its base change to $Y' \longrightarrow Y$. By lemma 1.9(c), f' is of finite type. Since Y' is noetherian, we can use valuative criterion to check properness of f'. The following diagram is self explainatory:





For second diagram, we used universal property of fibered products.

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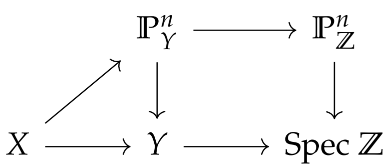
Theorem 2.7 (Main theorem of elimination theory). A projective morphism of noetherian schemes is proper.

Proof. Let $f : X \longrightarrow Y$ be a projective morphism. The we have



Here $X \longrightarrow \mathbb{P}^n_Y$ is closed immersion. By 2.6, it is sufficient to prove for $X = \mathbb{P}^n_{\mathbb{Z}}$ and Y=Spec $\mathbb{Z}$. Given,

$$\begin{array}{ccc} \text{Spec } K & \longrightarrow & \mathbb{P}^n_{\mathbb{Z}} \\ \downarrow & \nearrow & \downarrow \\ \text{Spec } R & \longrightarrow & \text{Spec } \mathbb{Z} \end{array}$$

Let z_1 be the point in Spec K and ξ_1 be the image of z_1 in X.

Existence of the lift: Cover $\mathbb{P}^n_{\mathbb{Z}}$ is covered by affine opens V_i=Spec $\mathbb{Z}[x_0/x_i, \ldots, x_n/x_i]$.

1. WLOG we can assume that $\xi_1 \in \cap_i V_i$. (We are using here induction on n with hypothesis that if $\xi_i \in \mathbb{P}^n$ then there exists a lift Spec $R \longrightarrow \mathbb{P}^n$. This is true for $n = 0$. If $\xi_i \in \mathbb{P}^n_{\mathbb{Z}} - V_i (\cong \mathbb{P}^{n-1}_{\mathbb{Z}})$ then we are done by induction hypothesis)
2. So $x_i/x_j \in \mathcal{O}_{\xi_1}$ are invertible. We have inclusion $k(\xi_1) \subseteq K$ given by morphism Spec $K \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ (1.10). Let f_{ij} be the image of x_i/x_j in K which are non-zero. Let $v : K^\times \longrightarrow G$ be associated valuation. Let $g_i = v(f_{i0})$ and g_k be minimal among them.
3. $f_{ik} \in R$ for each i since $v(f_{ik}) = g_i - g_k \geq 0$. Define homomorphism

$$\mathbb{Z}[x_0/x_i, \ldots, x_n/x_i] \longrightarrow R, \quad x_i/x_k \longmapsto f_{ik}$$

   which induces morphism Spec $R \longrightarrow V_i$ compatible with the morphism Spec $K \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ (since homomorphism is compatible with $k(\xi_i) \subseteq K$).

Uniqueness of the lift: Let f :Spec $R \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ be a lift. Since $\{V_i\}$ cover $\mathbb{P}^n_{\mathbb{Z}}$, $\{f^{-1}(V_i)\}$ cover Spec R. Since R is local, $f^{-1}(V_i)$=Spec R for some i. So f factors through V_i. Since f is compatible with Spec $R \longrightarrow V_i$, use above contruction to show f is the map constructed above.

If there are two liftings Spec $R \longrightarrow V_i$ and Spec $R \longrightarrow V_j$. Then the first diagram factors as second diagram ($A_i = \mathbb{Z}[x_0/x_i, \ldots, x_n/x_i]$)

$$\begin{array}{ccc} A_i & & \\ & \searrow & \\ & & R \\ & \nearrow & \\ A_j & & \end{array} \qquad \begin{array}{ccc} A_i & & \\ & \searrow & \\ & & A_i[(x_j/x_i)^{-1}] \longrightarrow R \\ & \nearrow & \\ A_j & & \end{array}$$

This is because $f_{ij}, f_{ji} = f_{ij}^{-1} \in R$ hence $f_{ij} \in R^\times$. Now note that Spec A_i and Spec A_j are patched along $A_i[(x_j/x_i)^{-1}] = A_j[(x_i/x_j)^{-1}]$. So both maps actually restrict to $V_i \cap V_j$ and are same map. □