Main theorem of Elimination theory

Ajay Prajapati

Dept. of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Under the guidance of
Dr. Anand Sawant

Elimination Theory

Resultant of two polynomials

$$f(X) = a_n X^n + \ldots + a_0,$$
$$g(X) = b_m X^m + \ldots + b_0$$

be two polynomials in $A[X]$ where A is an integral domain.
f and g have common root (in an algebraic closure containing A) $\iff$

$$det \begin{bmatrix} a_0 & \ldots & a_n & 0 & \ldots & 0 \\ 0 & a_0 & \ldots & a_{n-1} & a_n & \\ \vdots & & & & & \\ b_0 & \ldots & b_n & 0 & \ldots & 0 \\ 0 & b_0 & \ldots & b_{n-1} & b_n & 0 \\ \vdots & & & & & \end{bmatrix} = 0$$

Main theorem of Elimination theory

k-algebraically closed field

Another interpretation of resultants

Let $f, g \in k[y_1, \ldots, y_r][X]$. Then f and g define hypersurfaces H_f and H_g in $\mathbb{A}^r \times \mathbb{A}^1$. The previous slide says that projection of intersection of two hypersurfaces is again a hypersurface. If $\pi_1 : \mathbb{A}^r \times \mathbb{A}^1 \longrightarrow \mathbb{A}^r$ is the projection, then $\pi_1(H_f \cap H_g) =$zeros of resultant of f and g.

Main theorem of Elimination theory

k-algebraically closed field

Another interpretation of resultants

Let $f, g \in k[y_1, \dots, y_r][X]$. Then f and g define hypersurfaces H_f and H_g in $\mathbb{A}^r \times \mathbb{A}^1$. The previous slide says that projection of intersection of two hypersurfaces is again a hypersurface. If $\pi_1 : \mathbb{A}^r \times \mathbb{A}^1 \longrightarrow \mathbb{A}^r$ is the projection, then $\pi_1(H_f \cap H_g) =$zeros of resultant of f and g.

What about general algebraic sets? i.e. closed sets.
E.g.: $yX - 1 = 0$ in $\mathbb{A}^1 \times \mathbb{A}^1$ is closed and its projection on $\mathbb{A}^1$ is $\mathbb{A}^1 - \{0\}$.
Fix: Use projective coordinates in second component.

Main theorem of Elimination theory [Eisenbud, theorem 14.1]

If X is any variety over k, and Y is an Zariski closed subset of $X \times \mathbb{P}_k^n$, then image of Y under projection $X \times \mathbb{P}_k^n \longrightarrow X$ is closed.

An elementary proof can be found in the chapter 14 exercises of the book *Commutative Algebra with a view towards algebraic geometry*

Proj Construction

1. Analogue of affine space $\mathbb{A}^n_k$ is the scheme $\operatorname{Spec} k[x_1, ..., x_n]$ (again denoted by $\mathbb{A}^n_k$).

Proj construction

Let S be a graded ring. Let S_+ be the ideal $\bigoplus_{d>0} S_d$ (irrelevant ideal). Define set

$$\operatorname{Proj} S = \{\mathfrak{p} \subseteq S \text{ prime } | \mathfrak{p} \text{ homogenous and } \mathfrak{p} \not\supseteq S_+\}$$

If $\mathfrak{a}$ is a homogenous ideal, then $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj} S | \mathfrak{a} \subseteq \mathfrak{p}\}$. (sheaf of rings is also defined)

Proj Construction

1. Analogue of affine space $\mathbb{A}_k^n$ is the scheme $\operatorname{Spec} k[x_1, ..., x_n]$ (again denoted by $\mathbb{A}_k^n$).

Proj construction

Let S be a graded ring. Let S_+ be the ideal $\bigoplus_{d>0} S_d$ (irrelevant ideal). Define set

$$\operatorname{Proj} S = \{\mathfrak{p} \subseteq S \text{ prime } | \mathfrak{p} \text{ homogenous and } \mathfrak{p} \not\supseteq S_+\}$$

If $\mathfrak{a}$ is a homogenous ideal, then $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj} S | \mathfrak{a} \subseteq \mathfrak{p}\}$. (sheaf of rings is also defined)

Analog of $\mathbb{P}_k^n$ is the scheme $\operatorname{Proj} k[x_0, ...x_n]$ (again denoted by $\mathbb{P}_k^n$).

Proj Construction

1. Analogue of affine space $\mathbb{A}_k^n$ is the scheme $\operatorname{Spec} k[x_1, ..., x_n]$ (again denoted by $\mathbb{A}_k^n$).

Proj construction

Let S be a graded ring. Let S_+ be the ideal $\bigoplus_{d>0} S_d$ (irrelevant ideal). Define set

$$\operatorname{Proj} S = \{\mathfrak{p} \subseteq S \text{ prime } | \mathfrak{p} \text{ homogenous and } \mathfrak{p} \not\supseteq S_+\}$$

If $\mathfrak{a}$ is a homogenous ideal, then $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj} S | \mathfrak{a} \subseteq \mathfrak{p}\}$. (sheaf of rings is also defined)

Analog of $\mathbb{P}_k^n$ is the scheme $\operatorname{Proj} k[x_0, ...x_n]$ (again denoted by $\mathbb{P}_k^n$). $\mathbb{P}_k^n$ is covered by $n+1$ affine patches each isomorphic to $\mathbb{A}_k^n$ where i^{th} affine patch is $\operatorname{Spec} k[x_0/x_i, \ldots x_n/x_i]$.

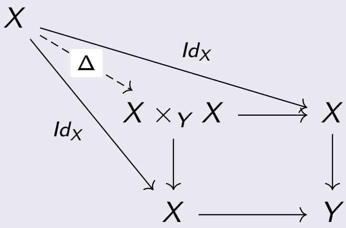
Seperatedness

Theorem

A topological space X is Hausdorff $\iff$ diagonal is closed in $X \times X$

Seperated morphism

A morphism of schemes $X \longrightarrow Y$ is said to be *seperated* if the image of diagonal map $\Delta(X)$ is closed.



- In usual euclidean topology, $\mathbb{P}^n(\mathbb{C})$ is compact while $\mathbb{A}^n(\mathbb{C})$ is not.

- In usual euclidean topology, $\mathbb{P}^n(\mathbb{C})$ is compact while $\mathbb{A}^n(\mathbb{C})$ is not.
- In general topology, X is compact $\iff X \times Z \longrightarrow Z$ is closed for all topological spaces Z.
- Relative version in topology: A map $X \longrightarrow Y$ is called **proper** if preimage of every compact set in Y is a compact set in X.

- In usual euclidean topology, $\mathbb{P}^n(\mathbb{C})$ is compact while $\mathbb{A}^n(\mathbb{C})$ is not.
- In general topology, X is compact $\iff X \times Z \longrightarrow Z$ is closed for all topological spaces Z.
- Relative version in topology: A map $X \longrightarrow Y$ is called **proper** if preimage of every compact set in Y is a compact set in X.
- **Example:** X is compact $\iff X \longrightarrow \{*\}$ is proper.
- If X and Y are Hausdorff then a map $X \longrightarrow Y$ is proper $\iff$ for all continuous maps $Z \longrightarrow Y$, the pullback $X \times_Y Z \longrightarrow Z$ is closed.

- In usual euclidean topology, $\mathbb{P}^n(\mathbb{C})$ is compact while $\mathbb{A}^n(\mathbb{C})$ is not.
- In general topology, X is compact $\iff$ $X \times Z \longrightarrow Z$ is closed for all topological spaces Z.
- Relative version in topology: A map $X \longrightarrow Y$ is called **proper** if preimage of every compact set in Y is a compact set in X.
- **Example:** X is compact $\iff$ $X \longrightarrow \{*\}$ is proper.
- If X and Y are Hausdorff then a map $X \longrightarrow Y$ is proper $\iff$ for all continuous maps $Z \longrightarrow Y$, the pullback $X \times_Y Z \longrightarrow Z$ is closed.

A morphism $X \longrightarrow Y$ is **universally closed** if base extension to morphism $Z \longrightarrow Y$ is closed for all such morphisms.

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

- In usual euclidean topology, $\mathbb{P}^n(\mathbb{C})$ is compact while $\mathbb{A}^n(\mathbb{C})$ is not.
- In general topology, X is compact $\iff$ $X \times Z \longrightarrow Z$ is closed for all topological spaces Z.
- Relative version in topology: A map $X \longrightarrow Y$ is called **proper** if preimage of every compact set in Y is a compact set in X.
- **Example:** X is compact $\iff$ $X \longrightarrow \{*\}$ is proper.
- If X and Y are Hausdorff then a map $X \longrightarrow Y$ is proper $\iff$ for all continuous maps $Z \longrightarrow Y$, the pullback $X \times_Y Z \longrightarrow Z$ is closed.

A morphism $X \longrightarrow Y$ is **universally closed** if base extension to morphism $Z \longrightarrow Y$ is closed for all such morphisms.

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

Proper morphisms (Grothendieck)

Seperated, finite type and universally closed.

Main theorem of Elimination theory

Closed immersions

A morphism $f : X \longrightarrow Y$ is called *closed immersion* if it is homeomorphism onto its image which is a closed subset of Y. (and some condition on sheaf of rings)

Main theorem of Elimination theory

Closed immersions

A morphism $f : X \longrightarrow Y$ is called *closed immersion* if it is homeomorphism onto its image which is a closed subset of Y. (and some condition on sheaf of rings)

A morphism $f : X \longrightarrow Y$ is called **projective morphism** if it factors as

$$\begin{array}{ccccc} & & Y \times \mathbb{P}^n_{\mathbb{Z}} & \longrightarrow & \mathbb{P}^n_{\mathbb{Z}} \\ & \overset{g}{\nearrow} & \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y & \longrightarrow & \text{Spec } \mathbb{Z} \end{array}$$

where g is a closed immersion.

Theorem, [Hartshorne, Theorem II.4.9]

A projective morphism of (noetherian) schemes is proper.

$$\begin{array}{ccccc} & & t(V)\times_{\operatorname{Spec}\mathbb{Z}}\mathbb{P}^n_{\mathbb{Z}} & \longrightarrow & \mathbb{P}^n_{\mathbb{Z}} \\ & \overset{g}{\nearrow} & \downarrow & & \downarrow \\ t(V)\times_{\operatorname{Spec} k}\mathbb{P}^n_k & \xrightarrow{\;f\;} & t(V) & \longrightarrow & \operatorname{Spec}\mathbb{Z} \end{array}$$

Valuative criterion of properness

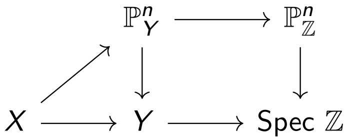
Let Y be a noetherian scheme, $f : X \longrightarrow Y$ be a morphism of finite type. Then f is proper $\iff$ for any valuation ring R with quotient field K, in the following commutative diagram

$$\begin{array}{ccc} \text{Spec } K & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \text{Spec } R & \longrightarrow & Y \end{array}$$

there is exactly one lifting Spec $R \longrightarrow X$ making whole diagram commute.

Corollary

- Closed immersions are proper.
- A composition of proper morphisms is proper.
- Proper morphisms are stable under base change.



Let R be any valuation ring with quotient field K and we have morphisms such that following diagram commutes:

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \mathbb{P}^n_{\mathbb{Z}} \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} \mathbb{Z} \end{array}$$