Some classical applications of Homology groups

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End-Semester Exam presentation

Outline of today's talk

- A theorem about homology of complements of embedded spheres and disks in a sphere.
- Jordan Curve theorem and its generalization
- Invariance of Domain
- A theorem about manifolds
- A theorem about division algebras over $\mathbb{R}$

First we compute homology groups of complements of embedded spheres and disks in a sphere.

Theorem

1. For an embedding $h : D^k \longrightarrow S^n$, $\tilde{H}_i(S^n - h(D^k)) = 0$ for all i.
2. For an embedding $h : S^k \longrightarrow S^n$ with $k < n$, then

$$\tilde{H}_i(S^n - h(S^k)) \cong \begin{cases} \mathbb{Z} & \text{for } i = n-k-1, \\ 0 & \text{otherwise} \end{cases}$$

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Let C be a simple closed curve in S^2. Then C seperates S^2 into two components.

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Jordan Curve Theorem

Let C be a simple closed curve in S^2. Then C seperates S^2 into two components.

(Brouwer) Above theorem generalises the Jordan Curve Theorem: A subspace of S^n homeomorphic to S^{n-1} seperates in into two components and these components has same homology groups as points.

Proof

Theorem

For an embedding $h : D^k \longrightarrow S^n$, $\tilde{H}_i(S^n - h(D^k)) = 0$ for all i.

Proof(a)

We use induction on k.

- **Base case:** For $k = 0$, $S^n - h(D^0)$ is homeomorphic to $\mathbb{R}^n$.
- For inductive hypothesis, we replace D^k by cube I^k. Let $A = S^n - h(I^{k-1} \times [0, 1/2])$ and $B = S^n - h(I^{k-1} \times [1/2, 1])$, so $A \cap B = S^n - h(I^k)$ and $X = \mathring{A} \cup \mathring{B} = A \cup B = S^n - h(I^{k-1} \times \{1/2\})$

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- **Induction Hypothesis:** $\tilde{H}_i(A \cup B) = 0$ for all i
- **Inductive Step:** Mayer-Vietoris sequence gives isomorphisms $\Phi_i : \tilde{H}_i(S^n - h(I^k)) \longrightarrow \tilde{H}_i(A) \oplus \tilde{H}_i(B)$ for all i.
- Modulo signs, the two components of Φ_i are induced by the inclusions $S^n - h(I^k) \hookrightarrow A$ and $S^n - h(I^k) \hookrightarrow B$.

Proof continued

Theorem

For an embedding $h : D^k \longrightarrow S^n$, $\tilde{H}_i(S^n - h(D^k)) = 0$ for all i.

Proof

- Suppose $\exists$ an i-dimensional cycle α of $S^n - h(I^k)$ which is not boundary in S^n-$h(I^k)$. Then α is not a boundary in atleast of A or B.

Proof continued

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- Suppose $\exists$ an i-dimensional cycle α of $S^n - h(I^k)$ which is not boundary in S^n-$h(I^k)$. Then α is not a boundary in atleast of A or B.
- By iteration, produce a nested sequence of of closed intervals $I_1 \supset I_2 \ldots$ in the last coordinate of I^k shrinking down to a point $p \in I$ such that α is not a boundary in $S^n - h(I^{k-1} \times I_m)$ for any m.

Proof continued

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- But α is the boundary of a chain β in $S^n - h(I^{k-1} \times \{p\})$.
- β is finite linear combination of singular simplicies with compact image in $S^n - h(I^{k-1} \times \{p\})$.

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- β is finite linear combination of singular simplicies with compact image in $S^n - h(I^{k-1} \times \{p\})$.
- Since $S^n - h(I^{k-1} \times I_m)$ forms increasing open cover of $S^n - h(I^{k-1} \times \{p\})$, by compactness, β is a chain of $S^n - h(I^{k-1} \times I_m)$ for some m.

Proof of (b)

Theorem

For an embedding $h : S^k \longrightarrow S^n$ with $k < n$, then

$$\tilde{H}_i(S^n - h(S^k)) \cong \begin{cases} \mathbb{Z} & \text{for } i = n-k-1, \\ 0 & \text{otherwise} \end{cases}$$

Proof

- When $k = 0$, then $S^n - h(S^0)$ is homotopic to S^{n-1}.
- Let $A = S^n - h(D^k_+)$ and $B = S^n - (D^k_-)$. $A \cap B = S^n - h(S^k)$ and $X = \mathring{A} \cup \mathring{B} = A \cup B = S^n - h(S^{k-1})$. By (a), we have $\tilde{H}_i(A) \cong \tilde{H}_i(B) \cong 0$.
- Apply Mayer-Vietoris with A and B, we have $\tilde{H}_i(S^n - h(S^k)) \cong \tilde{H}_{i+1}(S^n - h(S^{k-1}))$ for all i.

Consequences

- Applying previous proof to an embedding $h : S^n \longrightarrow S^n$, the Mayer-Vietoris sequence ends with

$$\tilde{H}_0(A) \oplus \tilde{H}_0(B) \longrightarrow \tilde{H}_0(S^n - h(S^{n-1})) \longrightarrow 0$$

- So $\tilde{H}_0(S^n - h(S^{n-1})) = 0$ which appears to contradict the fact that $S^n - h(S^{n-1})$ has two path components.

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- unless h is surjective $A \cap B = \varnothing$ and the Mayer-Vietoris sequence ends with $\tilde{H}_{-1}(\varnothing)$ which is $\mathbb{Z}$.
- So S^n cannot be embedded in $\mathbb{R}^n$
- More generally, there is no continuous injection $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ for $m > n$.

Alexander Horned Sphere

Jordan Schoenflies Theorem

Let C be a simple closed curve, then there is a homeomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ s.t. $f(C)$ is the unit circle in the plane.

This theorem is not true for $\mathbb{R}^3$ and a counterexample is provided by **Alexander Horned Sphere**.

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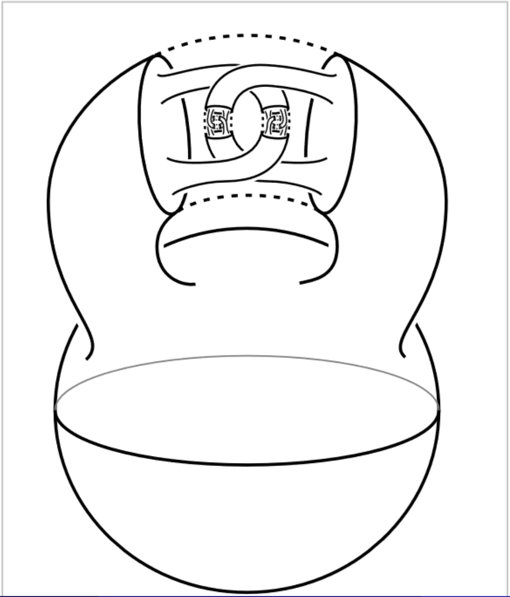
A subspace of S of $\mathbb{R}^3$ homeomorphic to S^2 such that unbounded component of $\mathbb{R}^3 - S$ is not simply connected.

Construction

- B_0-a ball, X_0-a solid torus obtained from B_0 by attaching $I \times D^2$ along $\partial I \times D^2$.
- To form $X_1 \subset X_0$, delete part of short handle so that what remains is a pair of linked handles attached to ball B_1 which is union of B_0 and two horns.
- X_n is a ball with 2^n handles attached and B_n is obtained from B_{n-1} by attaching 2^n horns.

Alexander Horned Sphere



Properties

- It can be proved that there is a map $f : B_0 \longrightarrow \mathbb{R}^3$ whose image is the previous figure.
- By compactness, f is homeomorphism onto its image, $B = f(B_0)$ and $S = f(\partial B_0)$ is the Alexander horned sphere.

Computation of $\pi_1(\mathbb{R}^3 - B)$

- $B = \cap_n X_n$, so $\mathbb{R}^3 - B = \cup_n (R^3 - X_n)$. Let $Y_n = R^3 - X_n$.
- It can be shown that $\pi_1(Y_n)$ is a free group on 2^n generators and $Y_n \hookrightarrow Y_{n+1}$ induces an injection $\pi_1(Y_n) \longrightarrow \pi_1(Y_{n+1})$.
- Suppose $\{\alpha_i^n\}$ are generators of $\pi_1(Y_n)$. Then $\alpha_i^n \longmapsto [\alpha_{2i+1}^{n+1}, \alpha_{2i+2}^{n+1}]$. So the map $\pi_1(Y_n) \longrightarrow \pi_1(Y_{n+1})$ is an injection $F_{2^n} \longrightarrow F_{2^{n+1}}$.
- $\pi_1(\mathbb{R}^3 - B) \cong \cup_n \pi_1(Y_n)$ by compactness argument: Each loop in $\mathbb{R}^3 - B$ lies in some Y_n.
- Clearly, $\pi_1(\mathbb{R}^3 - B)$ has trivial abelianization. In particular, it is a non-free group which is union of increasing sequence of free groups.

Theorem(**Invariance of Domain**) (Brouwer)

Let $U \subset \mathbb{R}^n$ be a open set, and $h : U \longrightarrow \mathbb{R}^n$ is an embedding, then the image $h(U)$ is an open set in $\mathbb{R}^n$.

Proof

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- Path components of $S^n - h(\partial D^n)$ are same as its components.
- The components of a space with finitely many components are open.
- $h(D^n - \partial D^n)$ is open in $S^n - h(\partial D^n)$ and hence also in S^n.

Manifold

Definition

M is called manifold if M is Hausdorff and is "locally Euclidean of dimension n" i.e. for each $x \in M$, $\exists$ nbhd U of x homeomorphic to an open subset of $\mathbb{R}^n$.

Equivalent definitions of a manifold

U can be taken to be homeomorphic to an open ball of $\mathbb{R}^n$ or to $\mathbb{R}^n$ itself.

Application to Manifolds

Theorem

Let M be a compact n-manifold and N be a connected n-manifold, then every injective map from M in N is homeomorphism.

Proof

- Suppose $h : M \longrightarrow N$ be an injective map. Clearly h is an embedding.
- Therefore suffices to prove that h is surjective. Since N is connected, suffices to prove that $h(M)$ is is both open and closed.
- $h(M)$ is closed.

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- $h(M)$ is closed.
- Use invariance of domain.

Division Algebras

Algebra

An **algebra** structure on $\mathbb{R}^n$ is a bilinear map $\mu : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Commutativity, associativity and identity element are not assumed.

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An algebra is a **division algebra** if the equations $ax = b$ and $xa = b$ has solution whenever $a \neq 0$.

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Frobenius Theorem (1877)

Upto isomorphism $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are the only finite dimensional associative division algebras over $\mathbb{R}$ with an identity element.

Some results on f.d. division algebras over $\mathbb{R}$

Theorem

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Any finite dimensional division algebra over $\mathbb{R}$ must have dimension 1, 2, 4 or 8.

Theorem

$\mathbb{R}$ and $\mathbb{C}$ are the only finite dimensional division algebras over $\mathbb{R}$ which are commutative and have an identity.

Proof

Proof

First assume that $\mathbb{R}^n$ has commutative.

- Define a map $f : S^{n-1} \longrightarrow S^{n-1}$, $x \longmapsto x^2/|x^2|$.
- f is continuous since the multiplication map and the norm map are continuous(?)

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- f is continuous since the multiplication map and the norm map are continuous(?)
- $f(x) = f(-x)$ so we have $\bar{f} : \mathbb{RP}^{n-1} \longrightarrow S^{n-1}$ which is injective: If $f(x) = f(y)$, then $x^2 = \alpha^2 y^2$.

$$x^2 - \alpha^2 y^2 = 0 \implies (x + \alpha y)(x - \alpha y) = 0 \implies x = \pm \alpha y$$

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$$x^2 - \alpha^2 y^2 = 0 \implies (x + \alpha y)(x - \alpha y) = 0 \implies x = \pm \alpha y$$

- Since $\mathbb{RP}^{n-1}$ and S^{n-1} are compact Hausdorff, $\bar{f}$ is an embedding.
- If $n \neq 1$, then $\bar{f}$ is surjective. Thus $\mathbb{RP}^{n-1} \cong S^{n-1} \implies n = 2$

Proof continued...

Proof

Suppose A is a 2-dimensional commutative division algebra with identity 1_A. We show $A \cong \mathbb{C}$.

- Let $j \notin \text{span}\{1_A\}$, then $\{1_A, j\}$ is a basis of A.
- WLOG, we can assume $j^2 = a.1_A$ where $a \in \mathbb{R}$.

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A finite dimensional commutative division algebra, not necessarily with an identity has dimension atmost 2. There do exists commutative division algebras without identity. E.g. $\mathbb{C}$ with $z.w = z.\bar{w}$

References

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