Milnor K Theory

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End-Semester Exam presentation

Previously...

Steinberg group of a ring R, $St_n(R)$ (For $n \geq 3$)

It is the group defined by the generators $x_{ij}(r)$, with $1 \leq i, j \leq n$, $i \neq j$ and $r \in R$ with the following relations:

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$$

$$[x_{ij}(r), x_{kl}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq l \\ x_{il}(rs) & \text{if } j = k \text{ and } i \neq l \\ x_{kj}(-sr) & \text{if } j \neq k \text{ and } i = l \end{cases}$$

The above relations are also satisfied by elementary matrices $e_{ij}(r)$ which generate the subgroup $E(R)$ of $GL(R)$, we have surjective group homomorphism $\varphi_n : St_n(R) \longrightarrow E_n(R),\ x_{ij}(r) \longmapsto e_{ij}(r)$. Taking direct limit, we get a map $\varphi : St(R) \longrightarrow E(R)$.

Previously...

K_2 of R

The group $K_2(R)$ is defined to be kernel of $\varphi : St(R) \longrightarrow E(R)$.

Theorem

(Steinberg) $K_2(R) = Z(St(R))$. In particular, $K_2(R)$ is abelian.

Thus $St(R)$ is a central extension of $E(R)$. Infact, we have

Theorem (Kervaire, Steinberg)

$St(R)$ is the universal central extension of $E(R)$.

Outline of today's talk

1. Steinberg Symbols
2. K_2 of fields
3. Some applications of K_2 in number theory
   - Hilbert Symbols and Quadratic Reciprocity
   - Brauer Groups
   - Norm residue symbols
4. Higher Milnor K groups

Motivation for Steinberg symbols

1. Suppose $x, y \in E(R)$ are s.t. $xy = yx$, then $[\varphi^{-1}(x), \varphi^{-1}(y)]$ is a well-defined element of $St(R)$
2. which maps to $[x, y] = 1$ under φ. i.e. $[\varphi^{-1}(x), \varphi^{-1}(y)] \in K_2(R)$.
3. Infact, this is the most useful way of constructing elements of $K_2(R)$ and in case of fields, $K_2(R)$ is generated by such elements.

Steinberg Symbols

Let R be a commutative ring.

Definition

Let $u, v \in R^\times$. The *Steinberg Symbol* $\{u, v\}$ is defined as

$$\{u, v\} := [\varphi^{-1}(d_{12}(u)), \varphi^{-1}(d_{13}(v))] \quad \text{where}$$

$$d_{12}(u) = \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d_{13}(v) = \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^{-1} \end{pmatrix}$$

Note that we have

- $d_{12}(u) = (e_{12}(u)e_{21}(-u^{-1})e_{12}(u))(e_{12}(1)e_{21}(-1)e_{12}(1))$ and
- $d_{13}(u) = (e_{13}(u)e_{31}(-u^{-1})e_{13}(u))(e_{13}(1)e_{31}(-1)e_{13}(1))$

Properties of Steinberg symbols

Notation

Define $w_{ij}(u) := x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u) \in St(R)$ and $h_{ij}(u) := w_{ij}(u)w_{ij}(-1) \in St(R)$. It is clear that $\{u, v\} = [h_{12}(u), h_{13}(v)]$.

Lemma

The Steinberg symbol map $R^\times \times R^\times \longrightarrow K_2(R)$ is skew-symmetric and bilinear. i.e. $\{u, v\} = \{v, u\}^{-1}$ and $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$.

Proof

Note that $\varphi(w_{23}(1))$ conjugates $d_{12}(u)$ to $d_{13}(u)$ and vice-versa. i.e.

$$\varphi(w_{23}(1))d_{12}(u)\varphi(w_{23}(1))^{-1} = d_{13}(u) \quad \text{since } \varphi(w_{23}(1)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Properties of Steinberg symbols

proof continued

Skew-symmetry is proved as

$$\begin{aligned}\{u,v\} &= [\varphi^{-1}(d_{12}(u)),\varphi^{-1}(d_{13}(v))]\\ &= [w_{23}(1)\varphi^{-1}(d_{13}(u))w_{23}(1)^{-1}, w_{23}(1)\varphi^{-1}(d_{12}(v))w_{23}(1)^{-1}]\\ &= w_{23}(1)[\varphi^{-1}(d_{13}(u)),\varphi^{-1}(d_{12}(v))]w_{23}(1)^{-1}\\ &= w_{23}(1)\{v,u\}^{-1}w_{23}(1)^{-1} = \{v,u\}^{-1} \quad \text{(since } K_2(R) \text{ is central)}\end{aligned}$$

Let $x_{ij}(u) = \varphi^{-1}(d_{ij}(u))$. Then

$$\begin{aligned}\{u, v_1v_2\} &= [x_{12}(u), x_{13}(v_1v_2)]\\ &= [x_{12}(u), x_{13}(v_1)x_{13}(v_2)]\\ &= [x_{12}(u), x_{13}(v_1)][x_{12}(u), x_{13}(v_2)][x_{13}(v_1),[x_{13}(v_2),x_{12}(u)]]\\ &= \{u,v_1\}\{u,v_2\}[x_{13}(v_1),\{u,v_2\}]^{-1}\\ &= \{u,v_1\}\{u,v_2\} \quad \text{(since } K_2(R) \text{ is central)}\end{aligned}$$

Lemma

Let R be any ring and $u, v \in R^\times$ and $i \neq j$, $k \neq l$, then the elements w_{ij} and h_{ij} of $St(R)$ defined above satisfy

$$(w_{ij}(u))^{-1} = w_{ij}(-u), \quad w_{ij}(u) = w_{ji}(-u^{-1}), \quad h_{ij}(1) = 1$$

$$w_{kl}(u)w_{ij}(v)(w_{kl}(u))^{-1} = \begin{cases} w_{ij}(v), & i,j,k,l \text{ all distinct}, \\ w_{lj}(-u^{-1}v), & k = i, \quad i,j,l \text{ all distinct}, \\ w_{il}(-vu), & k = j, \quad i,j,k \text{ all distinct}, \\ w_{ji}(-u^{-1}vu^{-1}), & k = i, j = l \\ w_{ji}(-uvu), & k = j, i = l \end{cases}$$

Corollary

Let $u, v \in R^\times$, then $h_{12}(uv) = h_{12}(u)h_{12}(v)\{u, v\}^{-1}$.

Properties of Steinberg symbols

Theorem

The Steinberg symbol map $R^\times \times R^\times \longrightarrow K_2(R)$ also satisfies

1. $\{u, -u\} = 1$ for $u \in R^\times$,
2. $\{u, 1-u\} = 1$ for $u \in R^\times$, $1-u \in R^\times$.

Proof

1. (a) By above corollary, we need to show $h_{12}(-u^2) = h_{12}(u)h_{12}(-u)$.
2. Using last identities of the lemma, we have

$$\begin{aligned} h_{12}(u)h_{12}(-u) &= w_{12}(u)w_{12}(-1)w_{12}(-u)w_{12}(-1) \\ &= w_{21}(u^{-2})w_{12}(-1) \\ &= w_{12}(-u^2)w_{12}(-1) = h_{12}(-u^2) \end{aligned}$$

3. (b) Since $-r = (1-r)/(1-r^{-1})$, the first part implies
$1 = \{r, -r\} = \{r, 1-r\}\{r, 1-r^{-1}\}^{-1} = \{r, 1-r\}\{r^{-1}, 1-r^{-1}\} = \{r, 1-$

Properties of Steinberg symbols

Corollary

If R is a finite field, then all Steinberg symbols vanish in $K_2(R)$.

Proof

1. Let $R = \mathbb{F}_q$. Then $\mathbb{F}_q^\times$ is cyclic say generated by u.
2. By bilinearity of symbol, it suffices to prove that $\{u, u\} = 1$.
3. If $char(R) = 2$, then $1 = -1$ and we have $\{u, u\} = \{u, -u\} = 1$.
4. Otherwise q is odd. By skew-symmetry, we have $\{u, u\} = \{u, u\}^{-1}$. i.e. $\{u, u\}$ has order atmost 2.
5. For any odd $m, n \in \mathbb{Z}$, we have $\{u, u\} = \{u, u\}^{mn} = \{u^m, u^n\}$.
6. Since odd powers of u are same as non-squares in $F_q^\times$, it suffices to find a non-square x s.t. $1 - x$ is also a non-square.
7. such an x exists because the map $x \longrightarrow 1 - x$ is an involution on the set $F_q - \{0, 1\}$ and this set consists of $(q-1)/2$ non-squares but only $(q-3)/2$ squares.

K_2 of fields

Theorem

If F is a field, then $K_2(F)$ is generated by Steinberg symbols.

Matsumoto's Theorem

If F is any (commutative) field, then $K_2(F)$ is the free abelian group on generators $\{u, v\}$, $u, v \in F^\times$, subject only to the relations of bilinearity in both variables and the relation $\{u, 1-u\} = 1$.

Some applications of K_2 in number theory

1. We have seen $K_2(\mathbb{F}_q) = 1$ for any finite field $\mathbb{F}_q$ while it is a classical fact that there are no non-commutative finite division algebras (Wedderburn Theorem).
2. Some close relationship between $K_2(F)$ for a field F and the existence of non-commutative finite dimensional division algebras over F. This is measured by *Brauer Group* $Br(F)$ and is an important invariant of the arithmetic of a field.

Hilbert symbol

Let F be a field of characteristic $\neq 2$. The *Hilbert symbol of F* is the map $(\ ,\ )_F : F^\times \times F^\times \longrightarrow \{\pm 1\}$ defined as: for $a, b \in F^\times$, $(a, b) = 1$ if there exists $x, y, z \in F$, not all zero s.t. $z^2 - ax^2 - by^2 = 0$, and $(a, b) = -1$ otherwise.

Why Hilbert Symbol?

Quadratic residue

Let $a \in \mathbb{Z} - \{0\}$ and p be an odd prime. If the equation $x^2 \equiv a \bmod p$ has solution in $F_p^{\times}$, then a is said to be *quadratic residue* mod p.

Legendre Symbol

$\left(\frac{a}{p}\right) = 1$ if a is a quadratic residue mod p and -1 o.w.

Quadratic reciprocity law (Theorema "aureum")

Let p and q be odd primes. Then

p is quadratic residue mod $q \iff q$ is quadratic residue mod p (if $p, q \not\equiv 3 \bmod 4$)

p is quadratic residue mod $q \iff q$ is quadratic non-residue mod p

More compactly, $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\epsilon(p)\epsilon(q)}$ where $\epsilon(n) = (n-1)/2$.

Properties of Hilbert Symbol

Proposition

Let F be a field of characteristic $\neq 2$ and suppose for any quadratic extension $F(\sqrt{q})$ of F, $N(F((\sqrt{q}))^{\times})$ has index atmost 2 in $F^{\times}$. Then the Hilbert symbol $(a,b)_F$ depends only on the Steinberg symbol $\{a,b\} \in K_2(F)$, and defines a homomorphism $K_2(F) \longrightarrow \{\pm 1\}$.

Lemma

The Hilbert symbol $(a,b)_F = 1 \iff a$ lies in the image of the norm map $N : F(\sqrt{b})^{\times} \longrightarrow F^{\times}$.

Proof

$(\implies)$ Let $z^2 = ax^2 + by^2$ where not all x, y, z are 0. If $x = 0$ then b is perfect square and $F(\sqrt{b}) = F$. If not then $N(z/x + \sqrt{b}y/x) = a$.
$(\impliedby)$ If $b = c^2$ then $(0, 1, c)$ is the solution of $z^2 = ax^2 + by^2$. If not then $a = N(\alpha + \sqrt{b}\beta) = \alpha^2 - b\beta^2 \implies (1, \beta, \alpha)$ is a solution.

Proof

1. By Matsumoto's theorem, it is sufficient to prove that $(a, b)_F$ bilinear in both variables and $(a, 1-a)_F = 1$ for all $a \in F - \{0, 1\}$ (which is obvious since $a1^2 + (1-a)1^2 = 1$).
2. Hilbert symbol is symmetric as it takes values $\{\pm 1\}$ hence sufficient to prove bilinearity in first variable.
3. If $(a_1, b)_F = (a_2, b)_F = 1$, then $a_1, a_2 \in Im(N) \implies a_1 a_2 \in Im(N)$ hence $(a_1 a_2, b)_F = 1$.
4. Similarly, if $(a_1, b)_F = 1$ and $(a_2, b)_F = -1$ then also result is clear.
5. Lastly, if $(a_1, b)_F = (a_2, b)_F = -1$, then both a_1, a_2 represent non-trivial element of the quotient $F^\times / N(F((\sqrt{q}))^\times)$ which has cardinality atmost 2. Hence $a_1 a_2 \in Im(N)$.

Local fields satisfy previous theorem

Definition

A field which is locally compact w.r.t. a non-discrete topology is called a *local field*.

Theorem

Any local field is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$, or a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, the field of formal Laurent power series over the finite field $\mathbb{F}_p$.

Theorem

Let F be a local field of characteristic $\neq 2$. Then for any non-trivial quadratic extension $F(\sqrt{b})$ of F, $N(F(\sqrt{q})^\times)$ has index exactly 2 in $F^\times$.

p-adic fields

Fix a positive prime $p \in \mathbb{Z}$.

p-adic metric

For $x \in \mathbb{Z}$, write $x = p^n y$ where $p \nmid y$. Define $|x|_p := p^{-n}$ is called p-adic norm on $\mathbb{Z}$. This can be extended to $\mathbb{Q}$ by $\left|\frac{a}{b}\right|_p = \frac{|a|_p}{|b|_p}$ and will induce a metric $d(x, y) = |x - y|_p$ for $x, y \in \mathbb{Q}$.

It satisfies **Strong triangle inequality**: $|x - y|_p \leq max\{|x|_p, |y|_p\}$.

p-adic numbers

The completion of $\mathbb{Q}$ w.r.t. $|.|_p$ is called p-adic numbers denoted by $\mathbb{Q}_p$.

Ostrowski's Theorem

Every non-trivial absolute value on $\mathbb{Q}$ is either equivalent to ususal absolute value or p-adic absolute value for some prime p.

Computation of $K_2(\mathbb{Q})$

Theorem

1. $K_2(\mathbb{Q})$ is a direct limit of finite abelian groups, and $K_2(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is an infinite direct sum of $\mathbb{Z}/2\mathbb{Z}$, one for each prime number p.
2. The Hilbert symbol $(\, , \,)_{\mathbb{Q}_p}$ when restricted to $\mathbb{Q}$, kills the summands of $K_2(\mathbb{Q})$ corresponding to primes other than p, and maps the summand corresponding to p onto $\{\pm 1\}$.
3. **(Hilbert Reciprocity law)** For $a, b \in \mathbb{Q}^\times$, we have
$$(a, b)_{\mathbb{R}} = \prod_{p \text{ prime}} (a, b)_{\mathbb{Q}_p}$$

Quadratic reciprocity law

Quadratic reciprocity

$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\epsilon(p)\epsilon(q)}$ where $\epsilon(n) = (n-1)/2$.

Proof

1. Suppose r is a prime $\neq p, q$, then $(p, q)_{\mathbb{Q}_r} = 1$
   (Chevalley's Theorem) If $a, b, c \in \mathbb{Z}$, then $aX^2 + bY^2 + cZ^2 = 0$ has a non-trivial solution in $\mathbb{F}_p$ for every prime p and it lifts to $\mathbb{Q}_p$ if $p \nmid 2abc$.
2. $(p, q)_{\mathbb{R}} = 1$
3. $(p, q)_{\mathbb{Q}_q} = \left(\frac{p}{q}\right)$
   (Hensel's Lemma): Let $f(X) \in \mathbb{Z}[X]$ be a polynomial and $f(X) \equiv 0 \mod p$ has a solution $y \in \mathbb{F}_p$ s.t. $f'(y) \not\equiv 0 \mod p$. Then $\exists b \in \mathbb{Z}_p$ s.t. $y \equiv b \mod p$ and $f(b) = 0$.

Proof of Quadratic Reciprocity Law

proof continued

1. $(p,q)_{\mathbb{Q}_p} = \left(\frac{q}{p}\right)$.
2. $(p,q)_{\mathbb{Q}_2} = (-1)^{(p-1)(q-1)/4}$
   We need to prove that $(p,q)_{\mathbb{Q}_2} = -1$ if $p, q \equiv 3 \bmod 4$ and is 1 o.w.
   Suppose $p \equiv 1 \bmod 4$ then either $p \equiv 1 \bmod 8$ or $p \equiv 5 \bmod 8$.
   Then either p or $p + 4q$ is square in $\mathbb{Q}_2$.
   If $p, q \equiv 3 \bmod 4$ and let (x, y, z) be primitive solution of $z^2 - px^2 - qy^2 = 0$. Taking mod 4, we get that $x^2 + y^2 + z^2 \equiv 0 \bmod 4$. But $x^2, y^2, z^2 \equiv 0$ or $1 \bmod 4$. Only possible solution is $x, y, z \equiv 0 \bmod 4$.

Brauer Groups

Central Simple Algebras

Let F be a field. A finite dimensional F-algebra A(associative with unit) is called *central simple* if $Z(A) \cong F$ i.e. center of A is precisely F and A has no non-trivial two sided ideals, i.e. A is simple as a ring.

The Wedderburn structure theorem implies that any such algebra A is F-isomorphic to $M_n(D)$ for some $n \geq 1$ and some f.d. division algebra D with center F.

Definition

We call two central simple algebras A and B *stably isomorphic* if for some r, s, $M_r(A) \cong M_s(B)$.

Brauer Groups

Definition

The *Brauer group* of F, denoted $Br(F)$, is the set of isomorphism class of central simple F-algebras, with operation $\otimes_F$, identity $[F]$ and $[A]^{-1} := [A^{op}]$ since $A \otimes_F A^{op} \cong End_F(A)$ via identification $(a \otimes b^{op})(c) = acb$.

Since $A \cong M_{n_1}(D_1)$ and $B \cong M_{n_2}(D_2)$, A is stably isomorphic to $B \iff D_1 \cong D_2$. Thus each stable isomorphism class contains unique central division algebra.
It is clear that $Br(F)$ is actually the set of isomorphism classes of central division algebras over F.

Frobenius Theorem

Every finite dimensional division algebra over $\mathbb{R}$ is either isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

So $Br(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$. $(\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R}))$

Brauer Groups as Cohomology groups

Fix an algebraic closure $\bar{F}$ of F and let F_{sep} denote the seperable closure of F in $\bar{F}$.

Brauer groups are actually cohomology group of the absolute Galois group in disguise.

Theorem

There is an isomorphism

$$H^2(Gal(F_{sep}/F), F_{sep}^{\times}) \longrightarrow Br(F)$$

Theorem (Kummer)

Let n be a positive integer and F be a field with $char(F) = 0$ or $char(F) \nmid n$ and containing μ_n, the group of n^{th} roots of unity. Let K/F be a sufficiently large Galois extension, in particular for $K = F_{sep}$ then there is an isomorphism $\varphi : F^\times/(F^\times)^n \longrightarrow Hom(Gal(K/F), \mu_n)$ given by $\varphi(\sigma)(x) = \sigma(y)y^{-1}$ where $y \in K$ s.t. $y^n = x$.

Proof

Consider the short exact sequence of $Gal(K/F)$-modules

$$1 \longrightarrow \mu_n \longrightarrow K^\times \xrightarrow{x \mapsto x^n} K^\times \longrightarrow 1$$

This gives a long exact sequence of cohomology groups

$$H^0(G, \mu_n) = \mu_n \longrightarrow H^0(G, F^\times) = F^\times \xrightarrow{x \mapsto x^n} F^\times$$
$$\xrightarrow{\delta} H^1(G, \mu_n) = Hom(G, \mu_n) \longrightarrow H^1(G, F^\times) \longrightarrow \dots$$

Norm residue symbol

1. Let n be a positive integer and F be a field with $char(F) = 0$ or $char(F) \nmid n$ and containing μ_n. Let G denote the absolute Galois group of F, $Gal(F_{sep}/F)$.
2. Then there is a homomorphism called *norm residue symbol*, $K_2(F) \longrightarrow \{n\text{-torsion of } Br(F)\}$ defined as:
3. Identify $Br(F)$ with $H^2(G; F_{sep}^{\times})$ and view Kummer isomorphism φ as taking values in
$$Hom(G, \mu_n) \cong Hom(G, \mathbb{Z}/n\mathbb{Z}) = H^1(G, \mathbb{Z}/n\mathbb{Z})$$
4. let $\beta : H^1(G, \mathbb{Z}/n\mathbb{Z}) \longrightarrow H^2(G, \mathbb{Z})$ be the connecting map in the long exact cohomology sequence of short exact sequence of G-modules
$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

Propetry of norm residue symbol

1. Then for $\{u, v\} \in K_2(F)$,
$$\{u, v\} \longmapsto (u, v) := v_*(\beta \circ \varphi(u))$$
where we think v as giving the map of G-modules $\mathbb{Z} \longrightarrow F_{sep}^{\times}$, $1 \longmapsto v$.

Theorem

The map $\{u, v\} \longmapsto (u, v)$ is well defined. $(u, v) = 1 \iff v$ lies in the image of the norm map $N : F(u^{\frac{1}{n}}) \longrightarrow F^{\times}$. (This explains the name "norm residue symbol")

Another definition of norm residue symbol

Definition

Given $a, b \in F^\times$ and ζ primitive n^{th}-root of unity, define $A_\zeta(a,b)$ the associative algebra with unit generated by two elements x and y and relations $x^n = a$, $y^n = b$ and $xy = \zeta yx$.

It can be proved that $A_\zeta(a,b)$ is central simple and map $F^\times \times F^\times \longrightarrow Br(F)$ defines a map $K_2(F) \longrightarrow Br(F)$ which is norm residue map defined above.

Higher Milnor K groups

For a field F, consider the tensor algebra of the group $F^\times$

$$T(F^\times) = \mathbb{Z} \oplus F^\times \oplus (F^\times \otimes F^\times) \oplus (F^\times \otimes F^\times \otimes F^\times) \oplus \dots$$

Notation: $l(x)$ for the element of degree 1 in $T(F^\times)$ for $x \in F^\times$.

Definition

The graded ring $K_*^M(F)$ is defined to be the quotient of $T(F^\times)$ by the ideal generated by the homogenous elements $l(x) \otimes l(1-x)$ with $x \neq 0, 1$. *Milnor K-groups* are defined to be the subgroup of elements of degree n.

Notation: The image of $l(x_1) \otimes \dots \otimes l(x_n)$ in K_n^M will be denoted by $\{x_1, \dots, x_n\}$.

- We have $K_0^M(F) = \mathbb{Z}$ and $K_1^M(F) = F^\times$.
- By Matsumoto's theorem, we also have $K_2^M(F) = K_2(F)$, the elements $\{x, y\}$ being the usual Steinberg symbols.
- Matsumoto's theorem was original motivation of Milnor to define the $K-$groups as above and he hoped that his construction will give some insight into the definition of higher K-groups of general rings.

Some results of Higher Milnor K groups

1. $K_n^M(\mathbb{F}_q) = 0$ for all $n \geq 2$ because $K_2^M(\mathbb{F}_q) = 0$.
2. For fields, Milnor K-groups does not agree with Quillen's K-groups beyond K_2. There is a natural map $\lambda_n : K_n^M(F) \longrightarrow K_n(F)$ which fails to be injective for global fields(i.e. number fields or finite extension of $\mathbb{F}_p(t)$) and $n = 4$.
3. Milnor K groups plays a fundamental role in **higher class field theory**.

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