Homological Dimension

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Overview

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2. Homological Dimension Theory

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Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor between two abelian categories. Assume that $\mathcal{A}$ has enough projectives.

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Now we choose $F = - \otimes_R {}_R N_S$: **Mod-R** $\longrightarrow$ **Mod-S** in above definition and define

$$\mathrm{Tor}_n^R(M, N) := L_n F(M_R) = \mathrm{H}_n(P_\bullet \otimes_R {}_R N_S)$$

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Remark: We can also use projective resolution of M_R to compute $\mathrm{Ext}^n_R(M, N)$ (recall $\mathrm{Hom}_R(-, N_R)$ is contravariant). This follows from results in Weibel, section 2.7 (specifically theorem 2.7.6).

Tor and Ext computations

Calculation

Recall from last class that for any abelian group B

$$\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, B) = \begin{cases} B/mB & \text{if } n = 0 \\ B[m] = \{b \in B : mb = 0\} & \text{if } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

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To see this, use the projective resolution

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of $\mathbb{Z}/m\mathbb{Z}$ and now $\mathrm{Tor}_*(\mathbb{Z}/m\mathbb{Z}, B)$ is the homology of the complex $0 \longrightarrow B \xrightarrow{m} B \longrightarrow 0$.

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- By structure theorem of finitely generated abelian groups we can write

$$A \cong \mathbb{Z}^r \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_s\mathbb{Z}$$

for some integers $r, m_1, \cdots, m_s$.

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- $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}^r, -)$ vanishes for all $n \neq 0$.
- So $\mathrm{Tor}_n^{\mathbb{Z}}(A, B) \cong \mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/m_1\mathbb{Z}, B) \oplus \cdots \oplus \mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/m_s\mathbb{Z}, B)$.

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to see that for all $\mathbb{Z}/m\mathbb{Z}$-modules B we have

$$\operatorname{Tor}_n^{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}, B) = \begin{cases} B/dB & \text{if } n = 0 \\ \{b \in B : db = 0\}/(m/d)B & \text{if } n \text{ is odd, } n > 0 \\ \{b \in B : (m/d)b = 0\}/dB & \text{if } n \text{ is even, } n > 0 \end{cases}$$

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In particular, if $d^2 \mid m$ and take $B = \mathbb{Z}/d\mathbb{Z}$ then we get that

$$\mathrm{Tor}_n^{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z}/d\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z} \quad \text{for all } n.$$

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- In particular, if $d^2 \mid m$, then

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If no finite resolution exists, we set $pd(A)$, $id(A)$, or $fd(A)$ equal to ∞.

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- If $R = \mathbb{Z}/m\mathbb{Z}$ with some $p^2|m$ then R has both global and Tor dimension ∞.
- Since every projective module is flat, $fd(M) \leq pd(M)$ for every $M \in$ **Mod-R**.

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2. $\sup\{fd(R/J) : J$ is a right ideal of $R\}$
3. $\sup\{fd(B) : B$ is a left R-module$\}$
4. $\sup\{fd(R/I) : I$ is a left ideal of $R\}$
5. $\sup\{d : \mathrm{Tor}_d^R(A, B) \neq 0.$ for some R-modules $A, B\}$

This common number (possibly ∞) is called the Tor-dimension of R.

Examples

- Every field has global and Tor dimension 0.
- $R = \mathbb{Z}$ has both global and Tor dimension 1.
- If $R = \mathbb{Z}/m\mathbb{Z}$ with some $p^2|m$ then R has both global and Tor dimension ∞.
- Since every projective module is flat, $fd(M) \leq pd(M)$ for every $M \in$ **Mod-R**.
- For $R = \mathbb{Z}$, $fd(\mathbb{Q}) = 0$ whereas $pd(\mathbb{Q}) = 1$.

Preparation of proof of Global Dimension Theorem

Projective Dimension Lemma

The following are equivalent for a right R-module A:

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1. Clearly $(4) \implies (1) \implies (2) \implies (3)$.

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1. Clearly $(4) \implies (1) \implies (2) \implies (3)$.
2. By dimension shifting $\mathrm{Ext}^{d+1}(A, B) \cong \mathrm{Ext}^1(M_d, B)$.
3. M_d is projective iff $\mathrm{Ext}_R^1(M_d, B) = 0$ for all B.

Preparation of proof of Global Dimension Theorem

Injective Dimension Lemma

The following are equivalent for a right R-module B:

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Lemma

$B \in$ **Mod-R** is injective $\iff$ $\text{Ext}_R^1(R/I, B) = 0$ for all right ideals I.

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Lemma

$B \in$ **Mod-R** is injective $\iff \mathrm{Ext}_R^1(R/I, B) = 0$ for all right ideals I.

Proof

1. Apply $\mathrm{Hom}(-, B)$ to $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$, we see that

$$0 \longrightarrow \mathrm{Hom}(R/I, B) \longrightarrow \mathrm{Hom}(R, B) \longrightarrow \mathrm{Hom}(I, B) \longrightarrow \mathrm{Ext}^1(R/I, B) \longrightarrow 0$$

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is exact. Use Baer's criterion.

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[Global Dimension Theorem] The following numbers are the same:

1. $\sup\{id(B) : B \in \textbf{Mod-R}\}$
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by dimension shifting. This implies M is injective, contradiction.

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Flat Dimension Lemma

The following are equivalent for a right R-module A:

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4. If

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is a resolution with all $F_i \in$ **Flat-R**, then the syzygy M_d is also flat.

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Lemma

$B \in$ **R-Mod** is flat $\iff$ $\mathrm{Tor}_1^R(R/I, B) = 0$ for all right ideals I.

Noetherian Rings

Proposition

If R is right noetherian, then

- $fd(A) = pd(A)$ for every finitely generated R-module A.

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- Because R is noetherian, A is noetherian R-module and there is a resolution

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  where P_i are f.g. free R-modules and M is f.p.

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- M is flat R-module.
- **(Lemma)** Every finitely presented flat R-module is projective.

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Theorem

The following are equivalent for every ring R:

1. R is von Neumann regular.
2. R has Tor-dimension 0.
3. Every R-module is flat.
4. R/I is projective for every finitely generated ideal I.

Thank You