MTH613A: Rings and Modules

Homological Dimension

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§1. Introduction

Recall that in last lecture of this course we were studying left derived functors of a right exact functor and vice-versa, and then defined the Ext and Tor functors. This note will be in direct continuation of it. In section 2 we will start by recalling the definitions of left and right derived functors. Then we will do some computations of Ext and Tor groups which will be used in section 3. In section 3 we will define three natural notion of dimension of a module which will be used to define two notions of dimension, global and Tor, of a ring and prove their characterizing properties. We end this note by giving a characterization of rings of small dimension in section 4. The main reference for this note is [Wei95], sections 3.1, 3.2, 3.3, 4.1, and 4.2.

§2. Preliminaries and some calculations

Let us first recall the notions of left and right derived functors.

Definition 2.1. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor between two abelian categories. Assume that $\mathcal{A}$ has enough projectives. Then we define *left derived functors* L_nF of F as

$$L_nF(A) := \mathrm{H}_n(F(P_\bullet)) \quad \text{for all } n \geq 0$$

and for any projective resolution $P_. \longrightarrow A \longrightarrow 0$.

Now we choose $F = - \otimes_R {}_RN_S$: **Mod-R** $\longrightarrow$ **Mod-S** in above definition and define

$$\mathrm{Tor}_n^R(M, N) := L_nF(M_R) = \mathrm{H}_n(P_\bullet \otimes_R {}_RN_S)$$

for any projective resolution $P_\bullet \longrightarrow M \longrightarrow 0$ of M in **Mod-R**.

Definition 2.2. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor between two abelian categories. Assume that $\mathcal{A}$ has enough injectives. Then we define *right derived functors* R^nF of F as

$$R^nF(A) := \mathrm{H}^n(F(I^\bullet)) \quad \text{for all } n \geq 0$$

and for any injective resolution $0 \longrightarrow A \longrightarrow I^\bullet$.

Now we choose $F = \mathrm{Hom}_R(M_R, -)$: **Mod-R** $\longrightarrow$ **Ab** in above definition and define

$$\mathrm{Ext}_R^n(M, N) := R^n \mathrm{Hom}_R(M_R, -)(A) = \mathrm{H}^n(\mathrm{Hom}_R(M_R, I^\bullet))$$

for any injective resolution $0 \longrightarrow N \longrightarrow I^\bullet$ of N in **Mod-R**.

Remark 2.3. We can also use projective resolution of M_R to compute $\mathrm{Ext}_R^n(M, N)$ (recall $\mathrm{Hom}_R(-, N_R)$ is contravariant). This follows from results in Weibel, section 2.7 (specifically theorem 2.7.6). Its proof proceeds by construction of a double complex from injective and projective resolutions using tensor product.

Now we do some Tor and Ext computations for some choice of R, A, and B.

Example 2.1. Recall from last class that

$$\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, B) = \begin{cases} B/mB & \text{if } n = 0 \\ B[m] = \{b \in B : mb = 0\} & \text{if } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

To see this, use the resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}/m\mathbb{Z}$ and now $\mathrm{Tor}_*(\mathbb{Z}/m\mathbb{Z}, B)$ is the homology of the complex $0 \longrightarrow B \xrightarrow{m} B \longrightarrow 0$.

Now we recall a general fact about adjoints and limits which will be used in proof of proposition 2.5 (proof can be found in [ML13], section V.5).

Theorem 2.4. [Limit Adjoint Theorem] Let $L : \mathcal{A} \longrightarrow \mathcal{B}$ be left adjoint to a functor $R : \mathcal{B} \longrightarrow \mathcal{A}$, where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary categories. Then

1. L preserves all colimits (coproducts, direct limits, cokernels, etc.).
2. R preserves all limits (products, inverse limits, kernels, etc.).

Proposition 2.5. For all abelian groups A and B :

1. $\mathrm{Tor}_1^{\mathbb{Z}}(A, B)$ is a torsion abelian group.
2. $\mathrm{Tor}_n^{\mathbb{Z}}(A, B) = 0$ for $n \geq 2$.

Proof. Because tensor product is left adjoint of Hom functor it preserves direct limits. As direct limits are exact in **Mod-R** (see [Ati18], exercise 2.19) homology preserves direct limit hence so does Tor. Now any module is direct limits of its finitely generated submodules and direct limit of torsion group is again torsion, WLOG we can assume that A is finitely generated $\mathbb{Z}$-module. By structure theorem of finitely generated abelian groups we write

$$A \cong \mathbb{Z}^r \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_s\mathbb{Z}$$

for some integers $r, m_1, \cdots, m_s$. $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}^r, -)$ vanishes for all $n \neq 0$. So

$$\mathrm{Tor}_n^{\mathbb{Z}}(A, B) \cong \mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/m_1\mathbb{Z}, B) \oplus \cdots \oplus \mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/m_s\mathbb{Z}, B).$$

By computations in example 2.1 the result follows. □

Example 2.2. Take $R = \mathbb{Z}/m\mathbb{Z}$ and $A = \mathbb{Z}/d\mathbb{Z}$ with $d \mid m$, then we use the free resolution

$$\cdots \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{m/d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{\epsilon} \mathbb{Z}/d \longrightarrow 0$$

to see that for all $\mathbb{Z}/m\mathbb{Z}$-modules B we have

$$\operatorname{Tor}_n^{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}, B) = \begin{cases} B/dB & \text{if } n = 0 \\ \{b \in B : db = 0\}/(m/d)B & \text{if } n \text{ is odd, } n > 0 \\ \{b \in B : (m/d)b = 0\}/dB & \text{if } n \text{ is even, } n > 0 \end{cases}$$

In particular, if $d^2 \mid m$ and take $B = \mathbb{Z}/d\mathbb{Z}$ then we get that $\operatorname{Tor}_n^{\mathbb{Z}/m\mathbb{Z}}(B, B) = \mathbb{Z}/d\mathbb{Z}$.

Proposition 2.6. $\operatorname{Ext}^n_{\mathbb{Z}}(A, B) = 0$ for $n \geq 2$ and all abelian groups A, B.

Proof. Embed B in an injective abelian group I^0; the quotient I^1 is divisible, hence injective. Therefore, $\operatorname{Ext}^*(A, B)$ is the cohomology of $0 \longrightarrow \operatorname{Hom}(A, I^0) \longrightarrow \operatorname{Hom}(A, I^1) \longrightarrow 0$. □

Exercise 2.1. When $R = \mathbb{Z}/m\mathbb{Z}$ and $B = \mathbb{Z}/d\mathbb{Z}$ with $d \mid m$, show that

$$0 \longrightarrow \mathbb{Z}/d\mathbb{Z} \xrightarrow{i} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{m/d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{m/d} \cdots$$

is an infinite periodic injective resolution of B. Then compute the groups $\operatorname{Ext}^n_{\mathbb{Z}/m\mathbb{Z}}(A, \mathbb{Z}/d\mathbb{Z})$ in terms of $A^* = \operatorname{Hom}(A, \mathbb{Z}/m\mathbb{Z})$. In particular, show that if $d^2 \mid m$, then

$$\operatorname{Ext}^n_{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z}/d\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z} \quad \text{for all } n.$$

Proof. First we use Baer's criterion to prove that $R = \mathbb{Z}/m\mathbb{Z}$ is an injective R-module: Any ideal of R is of the form $e\mathbb{Z}/m\mathbb{Z}$. The map f in the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & e\mathbb{Z}/m\mathbb{Z} & \longrightarrow & \mathbb{Z}/m\mathbb{Z} \\ & & \downarrow f & \swarrow g & \\ & & \mathbb{Z}/m\mathbb{Z} & & \end{array}$$

will be determined by where it sends e. Now $\frac{m}{e} f(e) = 0$ in $\mathbb{Z}/m\mathbb{Z}$ hence $e \mid f(e)$. Now define g by $1 \longmapsto \frac{f(e)}{e}$ which clearly extends f.

Note that Ext groups are cohomology of the following chain complex:

$$0 \longrightarrow A^* \xrightarrow{d} A^* \xrightarrow{m/d} A^* \xrightarrow{d} \cdots$$

Like in example 2.2 we can deduce the following:

$$\operatorname{Ext}^n_{\mathbb{Z}/m\mathbb{Z}}(A, \mathbb{Z}/d\mathbb{Z}) = \begin{cases} \operatorname{Hom}(A, \mathbb{Z}/d\mathbb{Z}) & \text{if } n = 0 \\ \{f \in A^* : (m/d)f = 0\}/dA^* & \text{if } n \text{ is odd, } n > 0 \\ \{f \in A^* : df = 0\}/(m/d)A^* & \text{if } n \text{ is even, } n > 0 \end{cases}$$

Putting $A = \mathbb{Z}/d\mathbb{Z}$ and using that $A^*[m/d] = \mathrm{Hom}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})[m/d] = A^* = \frac{m}{d}\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$ and $dA^* = 0$, the last part follows. □

§3. Homological Dimension Theory

We clearly have three different notions of dimension of a module in terms of the length of its projective, injective, and flat resolution.

Definition 3.1. Let A be a right R-module.

- The *projective dimension* $pd(A)$ is the minimum integer n (if it exists) such that there is a resolution of A by projective modules $0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$.
- The *injective dimension* $id(A)$ is the minimum integer n (if it exists) such that there is a resolution of A by injective modules $0 \longrightarrow A \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^n \longrightarrow 0$.
- The *flat dimension* $fd(A)$ is the minimum integer n (if it exists) such that there is a resolution of A by flat modules $0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$.

If no finite resolution exists, we set $pd(A)$, $id(A)$, or $fd(A)$ equal to ∞.

Now we want to define meaningful notion of homological dimension of a ring. One way to do this is to take dimension of ring as a module over itself. But this does not give much information about the ring. Instead we define projective dimension of A to be $\sup\{pd(A) : A \in \textbf{Mod-R}\}$. This is more useful notion of dimension because for example if this is 0 then we know that all modules are projective. We have therorem 3.2 (resp. 3.3) which is used to define this (resp. Flat version) notion. We will give its proof at the end.

Theorem 3.2. (Global Dimension Thm) The following numbers are same for any ring R:

1. $\sup\{id(B) : B \in \textbf{Mod-R}\}$
2. $\sup\{pd(A) : A \in \textbf{Mod-R}\}$
3. $\sup\{pd(R/I) : I \text{ is a right ideal of } R\}$
4. $\sup\{d : \mathrm{Ext}^d_R(A, B) \neq 0 \text{ for some right modules } A, B\}$.

This common number (possibly ∞) is called the *(right) global dimension* of R, $r.gl.\dim(R)$.

Theorem 3.3. (Tor-Dimension Theorem) The following numbers are same for any ring R:

1. $\sup\{fd(A) : A \text{ is a right } R\text{-module}\}$
2. $\sup\{fd(R/J) : J \text{ is a right ideal of } R\}$
3. $\sup\{fd(B) : B \text{ is a left } R\text{-module}\}$
4. $\sup\{fd(R/I) : I \text{ is a left ideal of } R\}$

5. $\sup\{d : \text{Tor}_d^R(A, B) \neq 0.$ for some R-modules $A, B\}$

This common number (possibly ∞) is called the *Tor-dimension* of R.

The following are global and Tor dimension of some rings.

- Every field has global and Tor dimension 0.
- $R = \mathbb{Z}$ has both global and Tor dimension 1. This is clear from proposition 2.5 and 2.6.
- If $R = \mathbb{Z}/m\mathbb{Z}$ with some $p^2|m$ then R has both global and Tor dimension ∞. This is clear from example 2.2 and exercise 2.1.
- Since every projective module is flat every projective resolution is flat resolution and therefore $fd(A) \leq pd(A)$ for every $A \in$ **Mod-R**.
- For $R = \mathbb{Z}$, $fd(\mathbb{Q}) = 0$ whereas $pd(\mathbb{Q}) = 1$.

Proposition 3.4. [Dimension shifting] If $0 \longrightarrow M \longrightarrow P \longrightarrow A \longrightarrow 0$ is exact with P projective, show that $L_iF(A) \cong L_{i-1}F(M)$ for $i \geq 2$ and that $L_1F(A)$ is the kernel of $F(M) \rightarrow F(P)$. More generally, if

$$0 \longrightarrow M_m \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

is exact with the P_i projective, then $L_iF(A) \cong L_{i-m-1}F(M_m)$ for $i \geq m+2$.

Proof. If $m = 0$ then it is clear from the long exact sequence associated to this short exact sequence. For $m > 0$, split this exact sequence into $(m+1)$ short exact sequences and then apply the case $m = 0$ to each of them. At the end combine information to get the result. □

Lemma 3.5. [Projective Dimension Lemma] TFAE for a right R-module A:

1. $pd(A) \leq d$.
2. $\text{Ext}_R^n(A, B) = 0$ for all $n > d$ and all right R-modules B.
3. $\text{Ext}_R^{d+1}(A, B) = 0$ for all R-modules B.
4. If
$$0 \longrightarrow M_d \longrightarrow P_{d-1} \longrightarrow P_{d-2} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$
is any resolution with the P's projective, then the syzygy M_d is also projective.

Proof. Since $\text{Ext}^*(A, B)$ may be computed using a projective resolution of A, it is clear that (4) $\implies$ (1) $\implies$ (2) $\implies$ (3). If we are given a resolution of A as in (4), then $\text{Ext}^{d+1}(A, B) \cong \text{Ext}^1(M_d, B)$ by dimension shifting. Now M_d is projective iff $\text{Ext}^1(M_d, B) = 0$ for all B, so (3) implies (4). □

The proof of following two lemmas is similar to the above lemma.

Lemma 3.6. [Injective Dimension Lemma] TFAE for a right R-module B:

1. $id(A) \leq d$.
2. $\text{Ext}_R^n(A, B) = 0$ for all $n > d$ and all R-modules A.
3. $\text{Ext}_R^{d+1}(A, B) = 0$ for all R-modules A.
4. If $0 \longrightarrow B \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{d-1} \longrightarrow M^d \longrightarrow 0$ is any resolution with the E^i injective, then the syzygy M^d is also injective.

Lemma 3.7. [Flat Dimension Lemma] TFAE for a right R-module A:

1. $fd(A) \leq d$.
2. $\text{Tor}_n^R(A, B) = 0$ for all $n > d$ and all left R-modules B.
3. $\text{Tor}_{d+1}^R(A, B) = 0$ for all left R-modules B.
4. If $0 \longrightarrow M_d \longrightarrow F_{d-1} \longrightarrow F_{d-2} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$ is a resolution with all F_i's flat, then the syzygy M_d is also flat.

Lemma 3.8. $B \in$ **Mod-R** is injective $\iff$ $\text{Ext}_R^1(R/I, B) = 0$ for all right ideals I.

Proof. Apply $\text{Hom}(-, B)$ to $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$, we see that

$$0 \longrightarrow \text{Hom}(R/I, B) \longrightarrow \text{Hom}(R, B) \longrightarrow \text{Hom}(I, B) \longrightarrow \text{Ext}^1(R/I, B) \longrightarrow 0$$

is exact. By Baer's criterion, B is injective $\iff$ $\text{Hom}(R, B) \longrightarrow \text{Hom}(I, B)$ is surjective for all right ideals I. □

Proof. **(Global dimension theorem)** By projective and injective dimension lemmas, $\sup(2) = \sup(4) = \sup(1)$. Since $\sup(1) = \sup(2) \geq \sup(3)$, want to prove $\sup(3) \geq \sup(1)$. Suppose not and let $d = \sup(pd(R/I))$ and assume $id(B) > d$ for some $B \in$ **Mod-R**. For this B, choose a resolution $0 \longrightarrow B \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{d-1} \longrightarrow M \longrightarrow 0$ with the E's injective. Then by dimension shifting, for all right ideals I we have

$$0 = \text{Ext}_R^{d+1}(R/I, B) \cong \text{Ext}_R^1(R/I, M).$$

This implies that M is injective by lemma 3.8 contradiction to assumption that $id(B) > d$. □

The proof of Tor dimension theorem is almost identitical. For $B \in$ **R-Mod**, let $B^* = \text{Hom}(B, \mathbb{Q}/\mathbb{Z}) \in$ **Mod-R** denote the *Pontrygin dual* of B. The following is analogue of 3.8.

Lemma 3.9. The following are equivalent for every $B \in$ **R-Mod**:

1. B^* is flat R-module.

2. $B^* \in$ **Mod-R** is injective.
3. $I \otimes_R B \cong IB = \{x_1 b_1 + \cdots + x_n b_n : x_i \in I, b_i \in I\} \subset B$ for every right ideal I of R.
4. $\mathrm{Tor}_1^R(R/I, B) = 0$ for all right ideals I.

Proof. For every inclusion $A' \subset A$ in **Mod-R**, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(A, B^*) & \longrightarrow & \mathrm{Hom}(A', B^*) \\ \downarrow \cong & & \downarrow \cong \\ (A \otimes B)^* = \mathrm{Hom}(A \otimes B, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(A' \otimes B, \mathbb{Q}/\mathbb{Z}) = (A' \otimes B)^*. \end{array}$$

Now using the fact (easy to prove) that $f : B \longrightarrow C$ is injective $\iff$ the dual map $f^* : B^* \longrightarrow C^*$ is surjective, we get

$$\begin{aligned} B^* \text{ is injective } &\iff (A \otimes B)^* \to (A' \otimes B)^* \text{ is surjective for all } A' \subset A \\ &\iff A' \otimes B \to A \otimes B \text{ is injective for all } A' \subset A \end{aligned}$$

which is if and only if B is flat. For $((2) \iff (3))$ using Baer's criterion we get

$$\begin{aligned} B^* \text{ is injective } &\iff (R \otimes B)^* \longrightarrow (I \otimes B)^* \text{ is surjective for all } I \subset R \\ &\iff I \otimes B \longrightarrow R \otimes B \text{ is injective for all } I \\ &\iff I \otimes B \cong IB \text{ for all } I. \end{aligned}$$

$((3) \iff (4))$ This follows immediately from the exact sequence $0 \longrightarrow \mathrm{Tor}_1(R/I, B) \longrightarrow I \otimes B \longrightarrow B \longrightarrow B/IB \longrightarrow 0$. Note that $R/I \otimes_R B \cong B/IB$ for all right ideals I. □

Proof. **(Tor-dimension theorem)** The flat dimension lemma (3.7) shows that $\sup(5) = \sup(1) \geq \sup(2)$. The same lemma over R^{op} shows that $\sup(5) = \sup(3) \geq \sup(4)$. We may assume that $\sup(2) \leq \sup(4)$, that is, that $d = \sup\{fd(R/J) : J \text{ is a right ideal}\}$ is at most the supremum over left ideals. We are done unless d is finite and $fd(B) > d$ for some left R-module B. For this B, choose a resolution $0 \longrightarrow M \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow B \longrightarrow 0$ with the F_i flat. But then for all ideals J we have

$$0 = \mathrm{Tor}_{d+1}^R(R/J, B) = \mathrm{Tor}_1^R(R/J, M)$$

We saw in lemma that this implies that M is flat, contradicting $fd(B) > d$. □

Recall that $fd(\mathbb{Q}_{\mathbb{Z}}) = 0$ whereas $pd(\mathbb{Q}_{\mathbb{Z}}) = 1$. Now we see that if the module is finitely generated over a Noetherian ring then this is not possible. First we prove a lemma.

Lemma 3.10. Every finitely presented flat R-module is projective.

Proof. For any $A, M \in$ **R-Mod** there is a natural map

$$\sigma : A^* \otimes_R M \longrightarrow \mathrm{Hom}_R(M, A)^*, \quad f \otimes m \longmapsto (h \longmapsto f(h(m)))$$

Claim: This map is an isomorphism for every finitely presented module M and for all A.

Proof. First it is easy to see that σ is an isomorphism for $M = R$. Hence σ is an isomorphism for $M = R^n$ for all n (by additivity of tensor product and Hom functor in first variable). Because M is f.p. we have $R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$ exact for some m, n.

$$\begin{array}{ccccccc} A^* \otimes R^m & \longrightarrow & A^* \otimes R^n & \longrightarrow & A^* \otimes M & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \sigma & & \\ \mathrm{Hom}(R^m, A)^* & \longrightarrow & \mathrm{Hom}(R^n, A)^* & \longrightarrow & \mathrm{Hom}(M, A)^* & \longrightarrow & 0 \end{array}$$

The upper row is exact because $A^* \otimes -$ is right exact. The lower row is exact because contravariant $\mathrm{Hom}(-, A)$ is left exact and the Pontrygin dual $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is exact because $\mathbb{Q}/\mathbb{Z}$ is injective. Now σ is an isomorphism by the five lemma. □

Let M is a finitely presented R-module. Suppose we have a surjection $B \longrightarrow C$ in **R-Mod**. Then natural map $C^* \longrightarrow B^*$ is injection. Now if M is flat then top arrow of the square

$$\begin{array}{ccc} C^* \otimes_R M & \longrightarrow & B^* \otimes_R M \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}(M, C)^* & \longrightarrow & \mathrm{Hom}(M, B)^* \end{array}$$

is injective. Hence the bottom row is an injection which implies that $\mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, C)$ is surjective. Hence M is projective. □

Proposition 3.11. If R is right noetherian, then

- $fd(A) = pd(A)$ for every finitely generated R-module A.
- $\mathrm{Tor} - \dim(R) = r.gl.\dim(R)$.

Proof. Since we can compute $\mathrm{Tor} - \dim(R)$ and $r.gl.\dim(R)$ by the modules R/I, it suffices to prove (1). Since $fd(A) \leq pd(A)$, it suffices to prove that if $fd(A) = n$ then $pd(A) \leq n$. Because R is noetherian and A is f.g. it is noetherian R-module and there is a resolution

$$0 \longrightarrow M \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

where P_i are finitely generated free R-modules and M is finitely presented. Since $fd(A) = n$ by lemma 3.7(4), M is flat R-module. By lemma 3.10 M is projective and we are done. □

§4. Rings of Small Dimension

Now we give characterization of some rings of small dimension. Firstly rings of global dimension are exactly semi-simple rings:

Theorem 4.1. The following are equivalent for every ring R, where by "R module" we mean either left R-module or right R-module.

1. R is semi-simple.
2. R has (left and/or right) global dimension 0.
3. Every R-module is projective.
4. Every R-module is injective.
5. R is noetherian, and every R-module is flat.
6. R is noetherian and has Tor-dimension 0.

Proof. ((1) $\iff$ (3) $\iff$ (4)) This was proven in assignment. ((2) $\iff$ (3)) and ((5) $\iff$ (6)) follows from definition. (3) implies that every R-module is flat and (1) implies that R is noetherian since $R = \sum_{i=1}^{n} I_i$ is finite sum of simples. ((5) $\implies$ (1)) For $I \in Id_{l/r}(R)$, R/I is finitely presented and hence projective by lemma 3.10. This implies $R \longrightarrow R/I$ splits giving I as direct summand of R. □

Definition 4.2. A ring R is called *von Newmann Regular* if for every $a \in R$ there is an $x \in R$ such that $axa = a$.

Now we see that rings of Tor-dimension 0 are exactly von Neumann regular rings. But first we study their structure.

Proposition 4.3. If R is von Neumann regular and I is a finitely generated right ideal of R, then there is an idempotent e (an element with $e^2 = e$) such that $I = eR$. In particular, I is a projective R-module, because $R \cong I \oplus (1-e)R$.

Proof. Suppose first that $I = aR$ and that $axa = a$. It follows that $e = ax$ is idempotent and that $I = eR$. By induction on the number of generators of I, we may suppose that $I = aR + bR$ with $a \in I$ idempotent. Since $bR = abR + (1-a)bR$, we have $I = aR + cR$ for $c = (1-a)b$. If $cyc = c$, then $f = cy$ is idempotent and $af = a(1-a)by = 0$. As fa may not vanish, we consider $e = f(1-a)$. Then $e \in I$, $ae = 0 = ea$, and e is idempotent:

$$e^2 = f(1-a)f(1-a) = f(f-af)(1-a) = f^2(1-a) = f(1-a) = e.$$

Moreover, $eR = cR$ because $c = fc = ffc = f(1-a)fc = efc$. Finally, we claim that I equals $J = (a+e)R$. Since $a + e \in I$, we have $J \subseteq I$; the reverse inclusion follows from the observation that $a = (a+e)a \in J$ and $e = (a+\text{e})\, e \in J$. □

Remark 4.4. The converse of proposition 4.3 also holds: If every finitely generated right ideal I of R is generated by an idempotent (i.e., $R \cong I \oplus R/I$), then R is von Neumann regular.

Theorem 4.5. The following are equivalent for every ring R:

1. R is von Neumann regular.
2. R has Tor-dimension 0.
3. Every R-module is flat.
4. R/I is projective for every finitely generated ideal I.

Proof. ((2) $\iff$ (3)) by definition. If I is finitely generated ideal of R then R/I is finitely presented hence by lemma 3.10, R/I flat $\iff$ R/I projective $\iff$ $R \cong I \oplus R/I$ as R-modules. Thus we have ((3) $\implies$ (4) $\iff$ (1)). ((4) $\implies$ (2)) If I is any ideal of R then $I = \varinjlim_\alpha I_\alpha$ where each I_α is finitely generated. Since Tor commutes with direct limits, we have $\mathrm{Tor}^1(R/I, B) = \varinjlim_\alpha \mathrm{Tor}^1(R/I_\alpha, B) = 0$ for all modules B (since by assumption R/I_α is projective hence flat). Hence Tor-dimension of R is 0. □

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