Modular Forms

Hecke Operators

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Basic definitions

- The group $SL_2(\mathbb{Z})$ is called **modular group**. There are certain types of its subgroups which are of interest to us.

Principal congruence subgroup of Level N

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (modN) \right\} \tag{1}$$

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congruence subgroup of level N

A subgroup Γ of $SL_2(\mathbb{Z})$ s.t. $\Gamma(N) \subset \Gamma$.They are of finite index.

Examples

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (modN) \right\} \tag{2}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (modN) \right\} \tag{3}$$

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$$(f[\gamma]_k)(\tau) = (c\tau + d)^{-k} f(\gamma \cdot \tau) \tag{4}$$

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- Suppose f is holomorphic and weakly modular w.r.t Γ. Since (1, N; 0, 1) $\in \Gamma(N) \subset \Gamma$, there exists minimum h s.t. (1, h; 0, 1) $\in \Gamma$ so we have $f(\tau) = f(\tau + h)$. h is called **period** of Γ.
- The function $q_h : \tau \mapsto e^{\frac{2\pi i \tau}{h}}$ takes $\mathcal{H}$ to D' and is also periodic with period h. Corresponding to f, function $g : D' \to \mathbb{C}$ where g(q)=f(hlog(q_h)/(2πi)) is well defined and $f(\tau) = g(e^{\frac{2\pi i \tau}{h}})$.

- Since f was assumed to be holomorphic on $\mathcal{H}$, the composition with g is holomorphic on punctured disk. So $g(q_h)=\sum_{n\in\mathbb{Z}} a_n q_h^n$, $q_h \in D'$.

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Modular form of weight k w.r.t Γ

is a function $f : \mathcal{H} \to \mathbb{C}$ satisfying

1. f is holomorphic
2. f is weakly modular of weight k w.r.t Γ
3. **Holomorphy condition:** $f[\alpha]_k$ holomorphic at ∞ for all $\alpha \in SL_2(\mathbb{Z})$.

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- Above definition of a function being holomorphic at ∞ can be extended to the function being holomorphic at other cusps.
- Condition (3) says that a modular form is also holomorphic at its cusps. This is very important in the definition to make the space of modular forms a finite dimensional space.

Eisenstein series

Let k >2 be an even integer. Define **Eisenstein series of weight k**

$$G_k(\tau) = \sum_{(c,d)\in\mathbb{Z}^2-(0,0)} \frac{1}{(c\tau+d)^k} = \sum_{(c,d)}{}' \frac{1}{(c\tau+d)^k}, \tau \in \mathcal{H} \qquad (5)$$

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- G_k is easily seen to weakly modular w.r.t. $SL_2(Z)$ and satisfies holomorphy condition.
- Using Poission summation formula, we can write its Fourier series

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \text{ where } \sigma_{k-1}(n) = \sum_{m|n} m^{k-1} \tag{6}$$

Cusp forms of weight k w.r.t Γ

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The discriminant function

Let $g_2(\tau) = 60G_4(\tau)$ and $g_3(\tau) = 140G_6(\tau)$. The **discriminant function** is

$$\Delta : \mathcal{H} \to \mathbb{C} \text{ given by } \Delta(\tau) = (g_2(\tau))^3 - 27(g_3(\tau))^2 \tag{7}$$

- Then Δ is weakly modular of weight 12 and holomorphic on $\mathcal{H}$ and $a_0 = 0, a_1 = (2\pi)^{12}$. So $\Delta \in \mathcal{S}_{12}(SL_2(\mathbb{Z}))$.

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- Fourier coefficients of Δ have special significance in number theory.

$$\Delta(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) q^n, q = e^{2\pi i \tau} \tag{8}$$

The arithmetical function $\tau(n)$ is known as **Ramanujan's tau function**.

Ramanujan conjectures

Ramanujan made three conjectures about them:

1. $\tau(mn) = \tau(m)\tau(n)$ if (m, n)=1
2. $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$ for prime p and r >0
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The first two are proved by using theory of Hecke operators. The third one turned out to be very deep result and very difficult to prove. In 1971, Deligne showed it as a consequence of third Weil conjecture and in 1974, he proved it by proving the third Weil conjecture (Riemann Hypothesis for non-singular projective variety).

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- To find its dimension, we first define **modular curve** of Γ, $Y(\Gamma) = \Gamma \backslash \mathcal{H}$. Then make it a Riemann surface by defining compatible local charts.
- If a point in $\mathcal{H}$ has trivial stabilizer subgroup in $SL_2(\mathbb{Z})$, then its very to define local charts. But there are points in $\mathcal{H}$ with non-trivial stabilizer. Special care is required to define local charts around them. They are called **Elliptic points** and are finite in number.

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- It turns out that $Y(\Gamma)$ have finitely many points missing corresponing to cusps. After adjoining the cusps, it becomes **compact Riemann surface** $X(\Gamma) = Y(\Gamma) \cup (\Gamma \backslash (\mathbb{Q} \cup \{\infty\})) = \Gamma \backslash \mathcal{H}^*$ where $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$.

How to find dimensions of $\mathcal{M}_k(\Gamma)$

1. Then we define differentials of degree k on the compact Riemann surface $X(\Gamma)$ by taking a collection of differentials of degree k on local charts with certain compatibility conditions. It is easy to see that they form complex vector space and is denoted by $\Omega^{\otimes k}(X(\Gamma))$.

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3. It is easy to see that this function f is weakly modular of weight 2k w.r.t. Γ. It turns out that this function is meromorphic at all cusps. Such type of functions are called **Automorphic forms** of weight 2k w.r.t Γ and are denoted by $\mathcal{A}_{2k}(\Gamma)$.

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4. It is easy to see that $\mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k(\Gamma)$ are subspaces of $\mathcal{A}_k(\Gamma)$.
5. It turns out that for k even, $\mathcal{A}_k(\Gamma) \cong \Omega^{\otimes k/2}(X(\Gamma))$ as complex vector spaces. Now the images of $\mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k(\Gamma)$ under this isomorphism are subspaces of $\Omega^{\otimes k/2}(X(\Gamma))$. It turns out that these subspaces can be defined entirely in terms of **linear space** of **canonical divisors** on $X(\Gamma)$.

How to find dimensions of $\mathcal{M}_k(\Gamma)$

- Now we use Riemann-Roch theorem to find the dimension of linear space of these divisors. So we get the dimension formula of $\mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k(\Gamma)$ in terms of genus of surface $X(\Gamma)$, number of elliptic and cusp points.

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- We can compute the dimension of space of modular forms and cusp forms w.r.t groups $\Gamma_0(N), \Gamma_1(N)$ and $\Gamma(N)$ in terms of N and k by computing the number of elliptic and cusp points of these groups.
- E.g. For $SL_2(\mathbb{Z})$, the full modular group and even k$\geq$ 4,

$$dim(\mathcal{M}_k(SL_2(\mathbb{Z}))) = \begin{cases} \lfloor \frac{k}{12} \rfloor \text{ if k } \equiv 2mod(12) \\ \lfloor \frac{k}{12} \rfloor + 1otherwise \end{cases} \tag{9}$$

- (chapter 2 and 3 of book A first course in Modular forms by Fred Diamond, Jerry Shurman)

Hecke operators

- It turns out that basis of $\mathcal{M}_k(\Gamma)/\mathcal{S}_k(\Gamma)$ can be constructed explicitly. They are generalisation of the Eisenstein series that we have seen above. Therefore this quotient is known as **Eisenstein space** and is denoted by $\mathcal{E}_k(\Gamma)$.

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Diamond operator (Hecke operator of first type)

For $d \in (\mathbb{Z}/N\mathbb{Z})^*$ and $f \in \mathcal{M}_k(\Gamma_1(N))$ define

$$< d > f = f[\alpha]_k \text{ for any } \alpha = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \in \Gamma_0(N) \text{ with } \delta \equiv d(modN) \quad (10)$$

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- This can easily be checked to be well-defined. This basically says that group $\Gamma_0(N)$ acts on $\mathcal{M}_k(\Gamma_1(N))$ and action is completely determined by lower-right entry of matrix.

Definition

Let $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}$ be a Dirichlet character. Then χ-**eigenspace** of $\mathcal{M}_k(\Gamma_1(N))$

$$\mathcal{M}_k(N,\chi) = \{f \in \mathcal{M}_k(\Gamma_1(N)) : f[\gamma]_k = \chi(d_\gamma)f \forall \gamma \in \Gamma_0\} \qquad (11)$$

- It can be proved (using basic linear algebra) that

$$\mathcal{M}_k(\Gamma_1(N)) = \oplus_\chi \mathcal{M}_k(N,\chi) \qquad (12)$$

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- For each χ, define the operator on $\mathcal{M}_k(\Gamma_1)$

$$\pi_\chi = \frac{1}{\phi(N)} \sum_{d \in (\mathbb{Z}/N\mathbb{Z})^*} \chi(d)^{-1} < d > \quad (13)$$

- Prove $\pi_\chi^2 = \pi_\chi$, $\pi_\chi(\mathcal{M}_k(\Gamma_1)) \subset \mathcal{M}_k(N,\chi)$, $\pi_\chi = 1$ on $\mathcal{M}_k(N,\chi)$, $\sum_\chi \pi_\chi = 1$ and $\pi_\chi \pi_{\chi'} = 0$.

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- Clearly, $\mathcal{M}_k(N,\chi)$ is χ-eigenspace of the diamond operators.

$$\mathcal{M}_k(N,\chi) = \{f \in \mathcal{M}_k(\Gamma_1(N)) :< d > f = \chi(d) f \ \forall d \in (\mathbb{Z}/N\mathbb{Z})^*\} \tag{14}$$

Double coset operators

- To define Hecke operators of second type, we need to look at more general type of operators called **double coset operators**. The diamond operators are also special case of these operators but it can be defined without them.
- Clearly, $\Gamma_1, \Gamma_2 \leq GL_2(\mathbb{Q})^+$. For each $\alpha \in GL_2(\mathbb{Q})^+$, the set

$$\Gamma_1 \alpha \Gamma_2 = \{\gamma_1 \alpha \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\} \tag{15}$$

  is **double coset** in $GL_2(\mathbb{Q})^+$. The group Γ_1 acts on $\Gamma_1 \alpha \Gamma_2$ by left multiplication partitioning it into orbits $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$. We prove that orbit space is finite.

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Lemma 2

Let Γ_1, Γ_2 and α be as above. Let $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$. Then left multiplication by α map

$$\Gamma_2 \to \Gamma_1\alpha\Gamma_2 \qquad \text{given by} \qquad \gamma_2 \mapsto \alpha\gamma_2 \tag{16}$$

induces a natural bijection between coset space $\Gamma_3 \backslash \Gamma_2$ and orbit space $\Gamma_1 \backslash \Gamma_1\alpha\Gamma_2$.

- Any two congruence subgroups G_1 and G_2 of $SL_2(\mathbb{Z})$ are **commensurable** meaning that the indices $[G_1 : G_1 \cap G_2]$ and $[G_2 : G_1 \cap G_2]$ are finite. So by using lemma 1 and lemma 2 we get that orbit space $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$ is finite.

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- Extend the weight k operator to $GL_2(\mathbb{R})^+$: For $\gamma = (a, b; c, d)$

$$f[\gamma]_k = (ad - bc)^{k-1}(c\tau + d)^{-k} f(\gamma(\tau)) \tag{17}$$

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Definition

Let Γ_1, Γ_2 and α be as above. The **weight-k** $\Gamma_1\alpha\Gamma_2$ **operator** on $f \in \mathcal{M}_k(\Gamma_1)$ defined by

$$f[\Gamma_1\alpha\Gamma_2]_k = \sum_j f[\beta_j]_k \tag{18}$$

where $\{\beta_j\}$ are orbit representatives, i.e., $\Gamma_1\alpha\Gamma_2 = \cup_j \Gamma_1\beta_j$

The double coset operator is well defined

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- Every $\gamma \in GL_2^+(\mathbb{Q})$ satisfies $\gamma = \alpha\gamma'$ where $\alpha \in SL_2(\mathbb{Z})$ and $\gamma' = r(a, b; 0, d)$ with $r \in \mathbb{Q}^+$ and a, b, d relatively prime.

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- If $f \in \mathcal{M}_k(\Gamma)$ then $f[\alpha]_k$ has Fourier expansion. We can use above to show that $f[\gamma]_k$ also has Fourier expansion and if $f[\alpha]$ has constant term zero then so does $f[\gamma]_k$.

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The double coset operator

- takes modular forms w.r.t Γ_1 to modular forms w.r.t Γ_2.
- takes cusp forms w.r.t Γ_1 to cusp forms w.r.t Γ_2.

Hecke operators of second type

- To define diamond operators as double coset operators, take any $\alpha \in \Gamma_0(N)$ and set $\Gamma_1 = \Gamma_2 = \Gamma_1(N)$. Then $< d_\alpha >= [\Gamma_1 \alpha \Gamma_2]_k$.

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- For second type of Hecke operators, take again $\Gamma_1 = \Gamma_2 = \Gamma_1(N)$ but $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, p prime. This operator is denoted by T_p. Thus,

$$T_p : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N)) \text{ given by } T_p f = f[\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N) \tag{19}$$

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- By definition, any double coset operator is specified by orbit representatives of $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$ which are coset representatives for $\Gamma_2 \backslash \Gamma_3$ left multiplied by α (lemma 2). So if we compute the coset representatives of $\Gamma_2 \backslash \Gamma_3$, we get exact representation of T_p.

Proposition

Let N $\in \mathbb{Z}^+$, let $\Gamma_1, \Gamma_2 = \Gamma_1(N)$ and let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ where p is prime.

The operator $T_p = [\Gamma_1 \alpha \Gamma_2]_k$ on $\mathcal{M}_k(\Gamma_1(N))$ is given by

$$T_p f = \begin{cases} \sum_{j=0}^{p-1} = f[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}]_k & \text{p}|\text{N} \\ \sum_{j=0}^{p-1} = f[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}]_k + f[\begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]_k & \text{p}\nmid\text{N, mp-nN=1} \end{cases} \tag{20}$$

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Proposition

Let d and e be elements of $(\mathbb{Z}/N\mathbb{Z})^*$ and let p, q be primes. Then

- $\langle d \rangle T_p = T_p \langle d \rangle$
- $\langle e \rangle \langle d \rangle = \langle d \rangle \langle e \rangle$
- $T_p T_q = T_q T_p$

Extension of Hecke operators

- We can extend both types of operators to all positive integers n.

Extension of diamond operators

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- Note that T_p and T_q commutes for different primes p and q.

Extension of second type of Hecke operators

Set $T_1 = 1$. T_p is already defined for primes p. For primes powers, define inductively

$$T_{p^r} = T_p T_{p^{r-1}} - p^{k-1} < p > T_{p^{r-2}}, r \geq 2 \quad (21)$$

Now, T_n is easily defined using unique factorization of integers.

What next

- An inner product on $\mathcal{S}_k(\Gamma)$ called **Petterson inner product**.
- W.r.t above inner product, both types of Hecke operators $\{<n>, T_n : (n, N) = 1\}$ become self-adjoint.

Theorem

The space $\mathcal{S}_k(\Gamma_1(N))$ has an orthogonal basis of simultaneous eigenforms for the Hecke operators $\{<n>, T_n : (n, N) = 1\}$

Proposition

Let $f \in \mathcal{M}_k(N, \chi)$. Then f is a normalized eigenform if and only if its Fourier coefficients satisfy following

1. $a_1(f) = 1$
2. $a_{p^{r+1}}(f) = a_p(f)a_{p^r}(f) - \chi(p)p^{k-1}a_{p^{r-1}}(f)$ for prime p and r >0
3. $a_m(f)a_n(f) = a_{mn}(f)$ when (m, n)=1

Ramanujan Conjecture

1. $\tau(mn) = \tau(m)\tau(n)$ if (m, n)=1
2. $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$ for prime p and r $>$0
3. $|\tau(p)| \leq p^{11/2}$ for all primes p

Using above proposition we can prove Ramanujan first two conjectures about τ function.