The possibl images

l-adic representations and congruences for congruences of modular forms

Ajay Prajapati

Indian Institute of Science, Bangalore

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$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$$

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$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$$

and the associated Dirichlet series has the Euler product expansion

$$\sum_{n=1}^{\infty} \tau(n) n^{-s} = \prod_{n=1}^{\infty} \frac{1}{(1 - \tau(p)p^{-s} + p^{11-2s})}$$

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 $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$

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Congruences

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$$\begin{aligned} \tau(n) &\equiv \sigma_{11}(n) \pmod{2^{11}} & \text{if} \quad n \equiv 1 \pmod{8} \\ \tau(n) &\equiv 1217\sigma_{11}(n) \pmod{2^{13}} & \text{if} \quad n \equiv 3 \pmod{8} \\ \tau(n) &\equiv 1537\sigma_{11}(n) \pmod{2^{12}} & \text{if} \quad n \equiv 5 \pmod{8} \\ \tau(n) &\equiv 705\sigma_{11}(n) \pmod{2^{14}} & \text{if} \quad n \equiv 7 \pmod{8} \end{aligned}$$

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$$\begin{aligned} \tau(n) &\equiv n^{-610} \sigma_{1231}(n) \pmod{3^6} & \text{if} \quad n \equiv 1 \pmod{3} \\ \tau(n) &\equiv n^{-610} \sigma_{1231}(n) \pmod{3^7} & \text{if} \quad n \equiv 2 \pmod{3} \end{aligned}$$

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$$\tau(n) \equiv n^{-30} \sigma_{71}(n) \pmod{5^3}$$
 if $(n,5) = 1$

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 $\begin{aligned} \tau(n) &\equiv n\sigma_9(n) \pmod{7} \quad \text{if} \quad n \equiv 0, 1, 2, \text{ or } 4 \pmod{7} \\ \tau(n) &\equiv n\sigma_9(n) \pmod{7^2} \quad \text{if} \quad n \equiv 3, 5, \text{ or } 7 \pmod{7} \end{aligned}$

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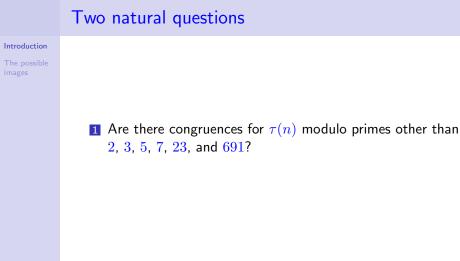
$$\begin{aligned} \tau(n) &\equiv 0 \pmod{23} & \text{if} \quad \left(\frac{p}{23}\right) = 1 \\ \tau(n) &\equiv 2 \pmod{23} & \text{if} \quad p = u^2 + 23v^2 \text{ for integers } u \neq 0, v \\ \tau(n) &\equiv -1 \pmod{23} & \text{for other } p \neq 23 \end{aligned}$$

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Two natural questions

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- 1 Are there congruences for $\tau(n)$ modulo primes other than 2, 3, 5, 7, 23, and 691?
- Are the congruences previously mentioned best possible or could one prove congruences modulo even higher powers?

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Two natural questions

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- 1 Are there congruences for $\tau(n)$ modulo primes other than 2, 3, 5, 7, 23, and 691?
- 2 Are the congruences previously mentioned best possible or could one prove congruences modulo even higher powers?
- 3 Are there similar congruences for fourier coefficients of other cusp forms?

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Let $f = \sum a_n q^n \in \mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$, and suppose

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Then there is a continuous homomorphism

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$$X^2 - a_p X + p^{k-1}$$

for each $p \neq \ell$.

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- Conditions on f are satisfied by unique cusp forms of weights 12, 16, 18, 20, 22, 26.
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Intuition: If the image of ρ_{ℓ} is small enough, a knowledge of the determinant of an element of the image will imply some ℓ -adic information about the trace of that element;

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3 Converse also holds.

The possible images

Lemma

Suppose that $\ell > 3$ and that $G \leq \operatorname{GL}_2(\mathbb{Z}_\ell)$ be a closed subgroup.

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Suppose that $\ell > 3$ and that $G \leq GL_2(\mathbb{Z}_\ell)$ be a closed subgroup. If the image of G under the map

 $(\mathrm{mod}\ \ell): \mathrm{GL}_2(\mathbb{Z}_\ell) \longrightarrow \mathrm{GL}_2(\mathbb{F}_\ell)$

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Proof (outline)

1 Let $G_n = \text{Image}(G \longrightarrow \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})).$

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3 H_2 is generated by three matrices $I + \ell u$ where $u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$.

Proof Outline

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In each case, $I + u \in SL_2(\mathbb{Z})$ hence $\exists \sigma \in G$ such that $\sigma \equiv I + u \pmod{\ell}$.

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 $\sigma^{\ell} = I + \ell (u + \ell v) + \ldots + (u + \ell v)^{\ell} \equiv I + \ell u \pmod{\ell^2}.$ because $u^2 = 0$ in each case.

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- 3 $I + \ell^{n-2}v \pmod{\ell^{n-1}}$ is in H_{n-1} . By induction hypothesis, $\exists \sigma \in G$ such that

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There are analogous results for $\ell = 2$ and $\ell = 3$, in which \mathbb{F}_{ℓ} is replaced by $\mathbb{Z}/8\mathbb{Z}$ or $\mathbb{Z}/9\mathbb{Z}$ respectively.

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Definition

 ℓ is called an exceptional prime for the cusp form f if the image of ρ_{ℓ} does not contain $\mathrm{SL}_2(\mathbb{Z}_{\ell})$.

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Corollary

Suppose that $\ell > 3$; then ℓ is exceptional for $f \iff$ the image of $\tilde{\rho_{\ell}}$ does not contain $\mathrm{SL}_2(\mathbb{F}_{\ell})$.

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Suppose that $\ell > 3$; then ℓ is exceptional for $f \iff$ the image of $\widetilde{\rho_{\ell}}$ does not contain $\mathrm{SL}_2(\mathbb{F}_\ell)$. For $\ell = 2$ or 3 this is still a sufficient condition for ℓ to be exceptional for f.

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$\operatorname{GL}_2(\mathbb{F}_\ell)$ acts on $V \cong \mathbb{F}_\ell^2$.

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A Borel subgroup of $\operatorname{GL}_2(\mathbb{F}_\ell)$ is any subgroup conjugate to the group of non-singular upper triangular matrices.

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A Cartan subgroup is a maximal semi-simple commutative subgroup.

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It is isomorphic to $(\mathbb{Z}/(\ell-1)\mathbb{Z})^2$.

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Introduction

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 $\operatorname{GL}_2(\mathbb{F}_\ell)$ acts on $V \cong \mathbb{F}_\ell^2$.

Definition

A Borel subgroup of ${\rm GL}_2(\mathbb{F}_\ell)$ is any subgroup conjugate to the group of non-singular upper triangular matrices.

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A Cartan subgroup is a maximal semi-simple commutative subgroup. For $\ell > 2$, a split Cartan subgroup is any subgroup conjugate to the group of non-singular diagonal matrices.

It is isomorphic to $(\mathbb{Z}/(\ell-1)\mathbb{Z})^2$.

Similarly, non-split Cartan subgroup is defined. It is isomorphic to $\mathbb{F}_{\ell^2}^\times.$

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Theorem

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Let G be a subgroup of $\operatorname{GL}_2(\mathbb{F}_{\ell})$. If ℓ divides the order of G, then either **1** G is contained in a Borel subgroup of $GL_2(\mathbb{F}_{\ell})$, or **2** $\operatorname{GL}_2(\mathbb{F}_\ell) \subset G$. If $(|G|, \ell) = 1$, let H be the image of G in $PGL_2(\mathbb{F}_{\ell})$; then **1** H is cyclic and G is contained in a Cartan subgroup, or **2** H is dihedral and G is contained in the normalizer of a Cartan subgroup but not in the Cartan subgroup itself, or

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Theorem

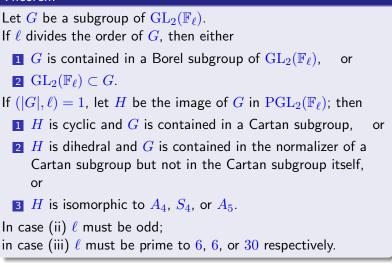
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Let G be a subgroup of \operatorname{GL}_2(\mathbb{F}_{\ell}).
If \ell divides the order of G, then either
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                                                                    or
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  2 H is dihedral and G is contained in the normalizer of a
     Cartan subgroup but not in the Cartan subgroup itself,
     or
  3 H is isomorphic to A_4, S_4, or A_5.
In case (ii) \ell must be odd;
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Corollary 1

Let $\rho_\ell: \operatorname{Gal}(K_\ell/\mathbb{Q}) \longrightarrow \operatorname{GL}_2(\mathbb{Z}_\ell)$ be any continuous homomorphism such that

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3 $H \cong S_4$.

Introduction

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Introduction

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Introduction

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- Any subgroup of a split Cartan subgroup is contained in a Borel subgroup.
- **2** Let C be a non-split Cartan subgroup. So C is cyclic of order $(\ell^2 1)$. Now $\tilde{\rho}_{\ell}$ factors through $\operatorname{Gal}(K_{\ell}^{ab}/\mathbb{Q}) \cong \mathbb{Z}_{\ell}^*$.
- 3 Since $(|Im(\mathbb{Z}_{\ell}^*)|, \ell) = 1$, $|Im(\mathbb{Z}_{\ell}^*)|| | (\ell 1)$.

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Introduction

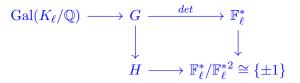
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Introduction

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$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$H \longrightarrow \mathbb{F}_{\ell}^{*}/\mathbb{F}_{\ell}^{*2} \cong \{\pm 1\}$$

Image of G in \mathbb{F}_{ℓ}^* consists of all $(k-1)^{th}$ powers and k is even.

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Corollary 2

Let $f = \sum a_n q^n \in \mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})$ be a normalized eigenform and ρ_ℓ be the Galois representation given by Serre-Deligne.

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2 $a_n \equiv 0 \pmod{\ell}$ whenever n is a quadratic non-residue $\pmod{\ell}$.

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Corollary 2

Let $f = \sum a_n q^n \in S_k(\operatorname{SL}_2(\mathbb{Z}), \mathbb{Z})$ be a normalized eigenform and ρ_ℓ be the Galois representation given by Serre-Deligne. Suppose that the image of $\tilde{\rho_\ell}$ does not contain $\operatorname{SL}_2(\mathbb{F}_\ell)$, so that ℓ is an exceptional prime for f. Then the three cases listed in Corollary 1 imply respectively the following congruences for the coefficients of f

1 There is an integer m such that

$$a_n \equiv n^m \sigma_{k-1-2m}(n) \pmod{\ell}$$

for all n prime to ℓ .

- 2 a_n ≡ 0 (mod ℓ) whenever n is a quadratic non-residue (mod ℓ).
- 3 $p^{1-k}a_p^2 \equiv 0, 1, 2, \text{ or } 4 \pmod{\ell}$ for all primes $p \neq \ell$.

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$$a_p \equiv p^m + p^{k-1-m} \pmod{\ell}$$

for $p \neq \ell$.

3 Let C be the Cartan subgroup and N its normalizer, and consider the homomorphism

$$\operatorname{Gal}(K_{\ell}/\mathbb{Q}) \longrightarrow N \longrightarrow N/C \cong \{\pm 1\}$$

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- We may distinguish (iii) from (ii) as follows: By an argument similar to used for (*ii*), the image of Frob(p) ∈ A₄ ⇐⇒ p is a quadratic residue (mod ℓ).
- 6 There are infinitely many p such that $\operatorname{Frob}(p)$ has order 4. For such p's, $p^{1-k}a_p^2 \equiv 2 \pmod{\ell}$.

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