Algebraic Geometry

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§I. Varieties

§§I.1. Affine Varieties

Exercise I.1.1. (a) Let Y be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

(c) Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

Solution. (a) $A(Y) = \frac{k[x,y]}{(y-x^2)} \cong k[x, x^2] = k[x]$.

(b) The coordinate ring of Z is

$$A(Z) = \frac{k[x, y]}{(xy - 1)}.$$

Suppose $\varphi : A(Z) \longrightarrow k[X]$ be a ring homomorphism. Since x is a unit in $A(Z)$, $\varphi(x)$ is a unit in $k[X]$ hence is in k. Then φ is not surjective. So it cannot be an isomorphism.

(c) Suppose that $f(x, y)$ is given by an irreducible quadratic polynomial

$$f(x, y) = ax^2 + hxy + by^2 + dx + ey + f$$

Since f is quadratic, degree 2 part is non-zero. Since k is algebraically closed, the degree 2 part can always be factorized as

$$ax^2 + hxy + by^2 = (a_1x + b_1y)(a_2x + b_2y)$$

where $(a_1x + b_1y)$ and $(a_2x + b_2y)$ are non-zero polynomials. Note that this implies that the linear part is non-zero (as f is irreducible). We have the following two cases:

Case 1: $(a_1x + b_1y)$ and $(a_2x + b_2y)$ are proportional. Then make a change of coordinates

$$(a_1x + b_1y) \longmapsto X, \qquad dx + ey + f \longmapsto Y$$

to obtain an equation of the form $Y = aX^2$ $(a \neq 0)$. Absorb the constant in Y to obtain the parabola equation $Y = X^2$. Note that the above change of coordinates is invertible i.e., $(a_1x + b_1y)$ and $dx + ey$ are non-proportional because if they were proportional then f would not be irreducible (as k is algebraically closed).

Case 2: $(a_1x + b_1y)$ and $(a_2x + b_2y)$ are non-proportional. Then make an invertible change of coordinates

$$(a_1x + b_1y) \longmapsto X, \qquad (a_2x + b_2y) \longmapsto Y$$

to obtain equation of the form $XY + aX + bY + c$ which can be written as

$$XY + aX + bY + c = (X - c_1)(Y - c_2) + c_3$$

Note here that since f is irreducible, $c_3 \neq 0$. Again make a linear change of coordinates $X - c_1 \longmapsto X$, $Y - c_2 \longmapsto c_3 Y$ to obtain equation of the standard hyperbola $XY = 1$.

Conclusion: Every conic over an algebraically closed field is either isomorphic to a parabola or a hyperbola according to whether degree 2 homogeneous part of the equation is respectively a square or non-square in $k[x, y]$. □

Exercise I.1.2. Let $Y \subseteq \mathbb{A}^3$ be the set

$$Y = \{(t, t^2, t^3) | t \in k\}.$$

Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

We say that Y is given by the **parametric representation** $x = t$, $y = t^2$, $z = t^3$.

Solution. It is easy to see that $Y = V(y - x^2, z - x^3)$. The ideal $I(Y) = (y - x^2, z - x^3)$ is a prime ideal. Hence Y is an affine variety. The coordinate ring of Y is

$$A(Y) = k[x, y, z]/(y - x^2, z - x^3) \cong k[x, x^2, x^3] = k[x].$$

So, by Proposition I.1.7 (which follows from the correspondence between prime ideals in $A(Y)$ and closed irreducible subsets of Y), $\dim(Y) = \dim(A(Y)) = 1$. □

Exercise I.1.3. Let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

Solution. Y is the locus of the solutions to the polynomial equations

$$xz - x = x(z - 1) = 0 \implies x = 0 \text{ or } z = 1 \quad \textbf{and} \quad x^2 - yz = 0$$

When $x = 0$, then the second equation becomes $yz = 0 \implies y = 0$ or $z = 0$. When $z = 1$, then the second equation becomes $x^2 - y = 0$. So, Y is a union of three components

$$x = 0, y = 0 \qquad x = 0, z = 0 \qquad z = 1, x^2 - y = 0$$

The corresponding prime ideals are $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$, and $\mathfrak{p}_3 = (z - 1, y - x^2)$. □

Exercise I.1.4. If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

Solution. Consider $\mathbb{A}^1 \times \mathbb{A}^1$ with the product topology. So the closed subsets of $\mathbb{A}^1 \times \mathbb{A}^1$ are a finite union of subsets the form $U \times V$ where $U, V \subset \mathbb{A}^1$ are closed. Now consider $\mathbb{A}^2$ with the Zariski topology. Then the set

$$V(xy - 1)$$

is closed in $\mathbb{A}^2$. But this is not closed in $\mathbb{A}^1 \times \mathbb{A}^1$: the only closed subsets of $\mathbb{A}^1$ are the finite set of points and the whole space $\mathbb{A}^1$. So the closed subsets of $\mathbb{A}^1 \times \mathbb{A}^1$ are a finite union of sets of the form

(a) finite subset of points

(b) finite union of vertical and horizontal lines

(c) the whole space

But $V(xy-1)$ cannot be written as finite union of sets of this forms. □

Exercise I.1.5. Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

Solution. Suppose X is an irreducible topological space and $U \subset X$ is a non-empty open subset. Suppose that U is not dense. So we can find another non-empty open subset V such that $U \cap V = \emptyset$. Then $U^c \cup V^c = X$ where U^c and V^c are proper closed subsets of X. Contradiction! So U is dense in X.

Suppose that $U = V_1 \cup V_2$ where V_1 and V_2 closed subsets of U. It is easy to see that this implies $\overline{U} = \overline{V_1} \cup \overline{V_2}$ where $\overline{V_1}$ and $\overline{V_2}$ denotes their closure in X. But $\overline{U} = X$ by above so irreducibility of X implies that either $\overline{V_1} = X$ or $\overline{V_2} = X$. But then either

$$V_1 = \overline{V_1} \cap U = U \quad \text{or} \quad V_2 = \overline{V_2} \cap U = U$$

(we are using that if V is closed in U then $\overline{V} \cap U = V$). So U is irreducible. □

Exercise I.1.6. Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume that $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r-1$. (See (7.1) for a generalization.)

Solution. Let $A = k[x_1, \dots, x_n]$ and $A(Y) = A/I(Y)$ be the affine coordinate ring of Y. Then by propsition I.1.7, $\dim Y = r$. By proposition I.1.13, $I(H) = (f)$ in A where f is a non-constant irreducible polynomial. The irreducible components of $Y \cap H$ corresponds to minimal primes belonging to $(\overline{f})$ in $A(Y)$ where $\overline{f}$ is the image of f in $A(Y)$. Note that because $Y \not\subseteq H$, $f \notin I(Y)$. So $\overline{f}$ is a non-zero divisor in $A(Y)$. By Krull's principal ideal theorem, every minimal prime belonging to $(\overline{f})$ has height 1. Let $\mathfrak{p}$ be such a prime and Z be the corresponding irreducible component of $Y \cap H$. Then by theorem I.1.8(b),

$$\begin{aligned}\dim B/\mathfrak{p} &= \dim B - \operatorname{height} \mathfrak{p} \\ &= 1\end{aligned}$$

Now $A(Z) = B/\mathfrak{p}$ and again applying proposition I.1.7, we get that

$$\dim Z = \dim A(Z) = \dim B/\mathfrak{p} = r - 1.$$

Hence every irreducible component of $Y \cap H$ has dimension $r-1$. □

§§I.2. Projective Varieties

§§I.3. Morphisms

Exercise I.3.1. A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism.

(a) For example, let

$$\varphi : \mathbb{A}^1 \longrightarrow \mathbb{A}^2, \qquad t \mapsto (t^2, t^3)$$

Show that φ defines a bijective bicontinuous morphism of $\mathbb{A}^1$ onto the curve $y^2 = x^3$, but that φ is not an isomorphism.

(b) For another example, let the characteristic of the base field k be $p > 0$, and define a map $\varphi : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ by $t \mapsto t^p$. Show that φ is bijective and bicontinuous but not an isomorphism. This is called the **Frobenius morphism.**

Solution. (a) **Injectivity:** Suppose $\varphi(t_1) = \varphi(t_2)$ then $t_1^2 = t_2^2$ and $t_1^3 = t_2^3$. Suppose that $t_1 = 0$ then $t_2 = 0 = t_1$. Similarly, when $t_2 = 0$ then $t_1 = 0 = t_2$. So can assume that $t_1 \neq 0$, $t_2 \neq 0$. Taking ratios, we get that $t_1 = t_2$.

Surjectivity: Suppose we are given a point (x, y) on the curve. Since k is algebraically closed, we can find a $t \in k$ such that $t^2 = x$. Putting this in the equation, we get that $y^2 = t^6$. If necessary, replacing t with $-t$, we get that $y = t^3$. So $(x, y) \in \text{im}(\varphi)$.

Since φ is defined by polynomials, it is continuous. More precisely, if V is a closed set in $\mathbb{A}^2$ defined by polynomials

$$f_1(x, y), \ldots, f_r(x, y)$$

then $\varphi^{-1}(V)$ is a closed set in $\mathbb{A}^1$ defined by polynomials $f_1(t^2, t^3), \ldots, f_r(t^2, t^3)$.

Also note that it is a closed map because $\varphi(\mathbb{A}^1)$ is the closed subset $V(y^2 - x^3)$ and $\varphi(finite\ sets) = finite\ sets$ which are closed in $\mathbb{A}^2$ (finite sets are the only proper closed subsets of $\mathbb{A}^1$). So φ is a bijective continous closed map, hence a homeomorphism. However, the inverse function $\psi : V(y^2 - x^3) \longrightarrow \mathbb{A}^1$ is given by

$$(x, y) \longmapsto \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ y/x & \text{otherwise} \end{cases}$$

To check that ψ is a morphism, we must verify that for every regular function f on $U \subset \mathbb{A}^1$ open, $f \circ \psi$ is regular on $\psi^{-1}(U)$. Let's take $f = \text{id}$ on $U = \mathbb{A}^1$ then $f \circ \psi = \psi$ on $V(y^2 - x^3)$. It is easy to see that around $(0, 0)$, ψ cannot be given by a ratio of two polynomials. So $f \circ \psi$ is not regular and hence ψ is not a morphism (of varieties). And φ is not an isomorphism.

(b) Here $\mathbb{A}^1 = k$. So the map φ is just

$$\varphi : k \longrightarrow k, \qquad t \longmapsto t^p$$

which is the regular p^{th}-power Frobenius homomorphism of k. Suppose that $x^p = y^p$ then $x^p - y^p = (x - y)^p = 0$ implying $x = y$ (Alternatively, since φ is a field homomorphism it is injective). Since k is algebraically closed, φ is surjective. Also the only closed sets of $\mathbb{A}^1$ are finite sets and $\mathbb{A}^1$ itself. Since

$$\varphi^{-1}(\mathbb{A}^1) = \mathbb{A}^1 \quad \text{and} \quad \varphi^{-1}(\mathit{finite\ set}) = \mathit{finite\ set},$$

φ is continuous. Again since

$$\varphi(\mathbb{A}^1) = \mathbb{A}^1 \quad \text{and} \quad \varphi(\mathit{finite\ sets}) = \mathit{finite\ sets}$$

φ^{-1} is continuous. So φ is a homeomorphism. Also φ induces a map of coordinate rings

$$k[x] \longmapsto k[x], \qquad f(x) \longmapsto f(x^p)$$

which is clearly not surjective (for example, $x \notin$ image). So φ cannot be an isomorphism. □

Exercise I.3.2. (a) Let $\varphi : X \longrightarrow Y$ be a morphism. Then for each $P \in X$, φ induces a homomorphism of local rings

$$\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \longrightarrow \mathcal{O}_{P,X}.$$

(b) Show that a morphism φ is an isomorphism $\iff$ φ is a homeomorphism, and the induced map φ_P^* on local rings is an isomorphism, for all $P \in X$.

(c) Show that if $\varphi(X)$ is dense in Y, then the map φ_P^* is injective for all $P \in X$.

Solution. (a) Let $\langle U, f\rangle$ be a representation of an element of $\mathcal{O}_{\varphi(P),Y}$ i.e., U is a neighbourhood of $\varphi(P)$ and $f : U \longrightarrow \mathbb{A}^1$ is a regular function. Then $\varphi_P^*([\langle U, f\rangle])$ is defined to be the equivalence class of $\langle \varphi^{-1}(U), f \circ \varphi\rangle$ in the local ring $\mathcal{O}_{P,X}$. This is a homomorphism of local rings because

$$\begin{aligned}
\varphi_P^*([\langle U,f\rangle][\langle V,g\rangle]) &= \varphi_P^*([\langle U,f\rangle\langle V,g\rangle]) \\
&= \varphi_P^*([\langle U\cap V, fg\rangle]) \\
&= [\langle \varphi^{-1}(U\cap V), (fg)\circ\varphi\rangle] \\
&= [\langle \varphi^{-1}(U)\cap\varphi^{-1}(V), (f\circ\varphi)(g\circ\varphi)\rangle] \\
&= [\langle \varphi^{-1}(U), f\circ\varphi\rangle][\langle \varphi^{-1}(V), g\circ\varphi\rangle] \\
&= \varphi_P^*([\langle U,f\rangle])\varphi_P^*([\langle V,g\rangle])
\end{aligned}$$

(b) ($\Longrightarrow$) Suppose that φ is an isomorphism with the inverse map ψ. Then φ is a homeomorphism. Also, it is clear from the definition of φ_P^* that it is functorial in nature: Suppose $X \xrightarrow{\mu} Y \xrightarrow{\lambda} Z$ are morphisms of varieties then for any $[\langle U, f\rangle]$ in $\mathcal{O}_{\lambda(\mu(P)),Z}$, we have

$$\begin{aligned}
(\lambda\circ\mu)_P^*([\langle U,f\rangle]) &= [\langle \mu^{-1}(\lambda^{-1}(U), f\circ(\lambda\circ\mu)\rangle] \\
&= [\langle \mu^{-1}(\lambda^{-1}(U)), (f\circ\lambda)\circ\mu\rangle] \\
&= \mu_P^*([\langle \lambda^{-1}(U), f\circ\lambda\rangle]) \\
&= \mu_P^*(\lambda_{\mu(P)}^*([\langle U,f\rangle])) \\
&= (\mu_P^*\circ\lambda_{\mu(P)}^*)([\langle U,f\rangle])
\end{aligned}$$

So $(\lambda \circ \mu)^*_P = \mu^*_P \circ \lambda^*_{\mu(P)}$. Applying this to our case, we get that $\varphi^*_P \circ \psi^*_{\varphi(P)} = (\psi \circ \varphi)^*_P = \mathrm{id}^*_P$. So φ^*_P is an isomorphism. This is true for any $P \in X$.

$(\Longleftarrow)$ Suppose that φ is a homeomorphism with the inverse map ψ and the induced map φ^*_P on local rings is an isomorphism, for all $P \in X$. Because of the functorial nature of φ^*_P, the inverse map $(\varphi^*_P)^{-1}$ must be given by $\psi^*_{\varphi(P)}$.

Now we check that φ is a morphism. Given any $f : U \longrightarrow \mathbb{A}^1$ a regular map on an open subset $U \subset Y$, we check that $f \circ \varphi : \varphi^{-1}(U) \longrightarrow \mathbb{A}^1$ is a regular map on $\varphi^{-1}(U) \subset X$. Let $P \in \varphi^{-1}(U)$. Then

$$\varphi^*_P([\langle U, f\rangle]) = [\langle \varphi^{-1}(U), f \circ \varphi\rangle] \in \mathcal{O}_{P,X}$$

So $f \circ \varphi$ is regular at P. This happens for each $P \in \varphi^{-1}(U)$. So $f \circ \varphi$ is regular. This happens for each regular function f on any open set U. So φ is a morphism.

Similarly, ψ can be checked to be a morphism using maps $\psi^*_{\varphi(P)}$. So φ is an isomorphism.

(c) Suppose that $[\langle U, f\rangle]$ and $[\langle V, g\rangle]$ be two elements of $\mathcal{O}_{\varphi(P),Y}$ such that

$$\varphi^*_P([\langle U, f\rangle]) = \varphi^*_P([\langle V, g\rangle]) \quad \text{i.e.,} \qquad [\langle \varphi^{-1}(U), f \circ \varphi\rangle] = [\langle \varphi^{-1}(V), g \circ \varphi\rangle]$$

This means that

$$f \circ \varphi = g \circ \varphi \quad \text{on} \quad \varphi^{-1}(U) \cap \varphi^{-1}(V) = \varphi^{-1}(U \cap V).$$

Since $U \cap V$ is open in X and $\varphi(X)$ is dense in Y, we have that $U \cap V \cap \varphi(X) \neq \emptyset$. So $\varphi^{-1}(U \cap V)$ is a non-empty open set. This means that $f = g$ on $\varphi(\varphi^{-1}(U \cap V))$ which is a non-empty set containing $\varphi(P)$. Now we claim that $\varphi(\varphi^{-1}(U \cap V))$ is dense in $U \cap V$.

Suppose not. Then there is an open set $W \subset U \cap V$ such that $\varphi(\varphi^{-1}(U \cap V)) \cap W = \emptyset$. Since $\varphi(\varphi^{-1}(U \cap V)) \subset U \cap V$, we have that $W \cap \varphi(X) = \emptyset$. As $W \subset X$ is also open, this is contradiction to densness of $\varphi(X)$. So $\varphi(\varphi^{-1}(U \cap V))$ is indeed dense in $U \cap V$. Since the set where $f = g$ is a dense subset of $U \cap V$, we get that $f = g$ on $U \cap V$. So $[\langle U, f\rangle] = [\langle V, g\rangle]$ and φ^*_P is injective. This happens for all $P \in X$. □

Exercise I.3.3. There are quasi-affine varieties which are not affine. For example, show that $X = \mathbb{A}^2 - \{(0,0)\}$ is not affine. [Hint: Show that $\mathcal{O}(X) \cong k[x,y]$ and use (3.5). See (III, Ex. 4.3) for another proof.]

Solution. $\mathbb{A}^2 - (0,0) = U_1 \cup U_2$ where $U_1 = \mathbb{A}^2 - \{x = 0\}$ and $U_2 = \mathbb{A}^2 - \{y = 0\}$ are open sets. Suppose f is a regular function on $\mathbb{A}^2 - (0,0)$. Then $f|_{U_1}$ and $f|_{U_2}$ are regular functions on their respective domains. But U_1 and U_2 are affine varieties in $\mathbb{A}^3$ with ideals $(xz-1) \subset k[x,y,z]$ and $(yz-1) \subset k[x,y,z]$. So their coordinate rings are

$$A(U_1) = k[x,y,z]/(xz-1) = k[x,1/x,y] \quad \text{and}$$
$$A(U_2) = k[x,y,z]/(yz-1) = k[x,y,1/y].$$

By theorem I.3.2, $f|_{U_1} \in A(U_1)$ and $f|_{U_2} \in A(U_2)$. So

$$f|_{U_1} = g_1(x,y)/x^n \quad \text{and} \quad f|_{U_2} = g_2(x,y)/y^m$$

where $g_1, g_2 \in k[x,y]$. Now, let P be any point in $\mathbb{A}^2 - \{(0,0)\}$. Since f is regular at P, there is a neighbourhood U of P such that

$$f|_U = g(x,y)/h(x,y)$$

where $g, h \in k[x,y]$ and h does not vanish at any point of U. Shrinking U, if necessary, we can assume that U is contained both in U_1 as well as U_2. Then

$$f|_U = g(x,y)/h(x,y) = g_1(x,y)/x^n = g_2(x,y)/y^m$$

WLOG, can assume that all these ratios are in lowest terms. The above equation gives $g_1(x,y)y^m = x^n g_2(x,y)$. If $m \neq 0$ then we have a contradiction as $y \nmid g_{(}x,y)$ and $y \nmid x$. So $m = 0$. Similarly $n = 0$ and $g_1(x,y) = g_2(x,y)$. So $f|_{U_1} = f|_{U_2} = g_1(x,y)$. Since $U_1 \cup U_2 = X$, we have that $f = g_1(x,y) \in k[x,y]$. Conversely, any element of $k[x,y]$ gives a regular function on X. So $\mathcal{O}(X) = k[x,y]$.

Suppose that X is affine. Then by theorem I.3.2(a), $A(X) = \mathcal{O}(X) = k[x,y]$. We have the inclusion $X \hookrightarrow \mathbb{A}^2$ which gives us a map of coordinate rings

$$A(\mathbb{A}^2) = k[x,y] \longrightarrow A(X) = k[x,y], f(x,y) \longmapsto f(x,y)|_X$$

This is actually an isomorphism of k-algebras. So by corollary I.3.7, $X \hookrightarrow \mathbb{A}^2$ is an isomorphism of varieties. Contradiction! So X is not affine. □

Exercise I.3.4. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties.

(a) Show that $X \times Y \subseteq \mathbb{A}^{n+m}$ with its induced topology is irreducible.

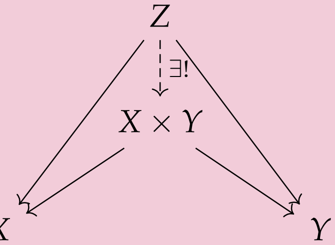
The affine variety $X \times Y$ is called the **product** of X and Y. Note that **its topology is in general not equal to the product topology** (Ex. 1.4).

(b) Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

(c) Show that $X \times Y$ is a product in the category of varieties, i.e., show

(i) the projections $p_1 : X \times Y \longrightarrow X$ and $p_2 : X \times Y \longrightarrow Y$ are morphisms, and

(ii) given a variety Z, and the morphisms $Z \longrightarrow X$, $Z \longrightarrow Y$, there is a unique morphism $Z \longrightarrow X \times Y$ making a commutative diagram



(d) Show that dim $X \times Y$ = dim X + dim Y.

Solution. (a) Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let

$$X_i = \{x \in X | x \times Y \subseteq Z_i\}, \quad i = 1, 2.$$

First we show that $X = X_1 \cup X_2$: Let $x \in X$ be any point. As Y is irreducible, $x \times Y$ which is isomorphic to Y is also irreducible. Let $W_i = (x \times Y) \cap Z_i$ for $i = 1, 2$. Then $W_1 \cup W_2 = x \times Y$. So by irreducibility, either $W_1 = x \times Y$ or $W_2 = x \times Y$. This means either $x \in X_1$ or $x \in X_2$.

Now we prove that X_1, X_2 are closed in X. Then the irreducibility of will imply that either $X = X_1$ or X_2. So $X \times Y = Z_1$ or Z_2 and hence $X \times Y$ is irreducible. Fix $y \in Y$. Then the map

$$\varphi : X \longrightarrow Y, \qquad x \longmapsto (x, y)$$

is a continuous map (since it is defined by polynomials). Now $X_i = \varphi^{-1}(Z_i)$. Since Z_i's are closed X_i's are closed in X as well.

Now we prove that $X \times Y$ is an algebraic set. Combining with above irreducibility result will give that $X \times Y$ is an affine variety. Let the coordinates of $\mathbb{A}^n$, $\mathbb{A}^m$, and $\mathbb{A}^{m+n}$ be given by $x_1, \dots, x_n$ and $y_1, \dots, y_m$ and $x_1, \dots, x_n, y_1, \dots, y_m$ respectively. Let

$$f_1(x_1, \dots, x_n), \dots, f_r(x_1, \dots, x_n)$$

generates the ideal $I(X)$ in $k[x_1, \dots, x_n]$ and

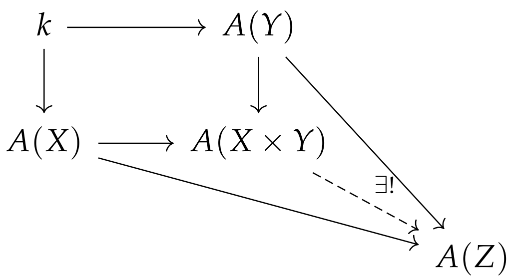
$$g_1(y_1, \dots, y_m), \dots, g_s(y_1, \dots, y_m)$$

generates the ideal $I(Y)$ in $k[y_1, \dots, y_m]$ then it is easy to see that

$$f_1(x_1, \dots, x_n), \dots, f_r(x_1, \dots, x_n), g_1(y_1, \dots, y_m), \dots, g_s(y_1, \dots, y_m)$$

generates the ideal of $X \times Y$ in $k[x_1, \dots, x_n, y_1, \dots, y_m]$.

(b) We will use the result of part (c) in this. It says that $X \times Y$ is a product in the category of affine varieties. Let Z be an affine variety. Then Theorem I.3.5 says that giving morphisms $Z \longrightarrow X$, $Z \longrightarrow Y$ is equivalent to giving k-algebra homomorphisms $A(X) \longrightarrow A(Z)$ and $A(Y) \longrightarrow A(Z)$. Universal property of $X \times Y$ gives us the following commutative diagram



Now tensor product is a coproduct in the category of k-algebras. So we should immediately say that $A(X \times Y) = A(X) \otimes_k A(Y)$. But we must be careful here as we are only working in the full subcategory of reduced finitely generated k-algebras as the following exercise (I.1.5) tells us

Exercise I.3.5. Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, $\iff$ B is a finitely generated k-algebra with no nilpotent elements.

Solution. ($\Longrightarrow$) Suppose that $B = k[x_1, \ldots, x_n]/I(Y)$ where Y is an affine algebraic set. Then it is easy to see that $I(Y)$ is a radical ideal of $k[x_1, \ldots, x_n]$ (also follows from corollary I.1.4). Therefore B has no nilpotent elements. It is easy to see that B is finitely generated k-algebra.

($\Longleftarrow$) Suppose that B is a finitely generated k-algebra with no nilpotent elements. Then $B = k[x_1, \ldots, x_n]/I$ where I is a radical ideal of $k[x_1, \ldots, x_n]$. Then by proposition I.1.2(d), $I(Y) = I$ where $Y = Z(I)$. So $A(Y) = B$. □

To say that $A(X \times Y) = A(X) \otimes_k A(Y)$, we must prove that tensor product is still the coproduct in this smaller category. It is clear that tensor product of two finitely generated k-algebras is again finitely generated. It is also true that tensor product of two reduced k-algebras is again reduced. So tensor product is still the coproduct in this smaller category.

(c) Given a regular function $f : U \longrightarrow \mathbb{A}^1$ on an open subset $U \subset Y$, we have

$$f \circ p_1 : U \times Y \longrightarrow \mathbb{A}^1, \quad (x, y) \longmapsto f(x)$$

which is clearly regular at every point of $U \times Y$. So p_1 is a morphism. Similarly, p_2 is a morphism.

Given morphisms $\varphi : Z \longrightarrow X$, $\psi : Z \longrightarrow Y$, we get a unique morphism $Z \longrightarrow X \times Y$ given by $z \longmapsto (\varphi(z), \psi(z))$ which makes the given diagram commutative.

(d) By proposition I.1.7,

$$\begin{aligned} \dim X \times Y &= \dim A(X \times Y) \\ &= \dim A(X) \otimes A(Y) \qquad \text{(by (b))} \\ &= \text{trans. deg}\, K(A(X) \otimes A(Y)) \qquad \text{(by theorem 1.8(a))} \\ &= \text{trans. deg}\, K(A(X)) + \text{trans. deg}\, K(A(Y)) \qquad (*) \\ &= \dim A(X) + \dim A(Y) \\ &= \dim X + \dim Y \end{aligned}$$

where equality in $(*)$ is as follows: Let

$$\begin{aligned} A(X) &= k[x_1, \ldots, x_n] = k[X_1, \ldots, X_n]/I(X) \quad \text{where } x_i = X_i \bmod I(X) \\ A(Y) &= k[y_1, \ldots, y_m] = k[Y_1, \ldots, Y_m]/I(Y) \quad \text{where } y_i = Y_i \bmod I(Y) \end{aligned}$$

be coordinate rings then $A(X) \otimes A(Y) \cong k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ and

$$K(A(X) \otimes A(Y)) = k(x_1, \ldots, x_n, y_1, \ldots, y_m) = k(x_1, \ldots, x_n)(y_1, \ldots, y_m).$$

So the transcendence degrees add up. □

Exercise I.3.6. Let $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ be a morphism of $\mathbb{A}^n$ to $\mathbb{A}^n$ given by n polynomials $f_1, \ldots, f_n$ of n variables $x_1, \ldots, x_n$. Let $J = \det |\partial f_i / \partial x_j|$ be the Jacobian polynomial of φ.

If φ is an isomorphism (in which case we call φ an automorphism of $\mathbb{A}^n$) show that J is a nonzero constant polynomial.

Solution. Let ψ be the inverse morphism of φ. We have canonical i^{th}-component projection morphism $\rho_i : \mathbb{A}^n \longrightarrow \mathbb{A}^1$. Then $\rho_i \circ \psi : \mathbb{A}^n \longrightarrow \mathbb{A}^1$ is a morphism which is same as a global regular function. Since $\mathbb{A}^n$ is affine, by theorem I.3.2(a), $\rho_i \circ \psi$ must be a polynomial in $k[x_1, \ldots, x_n]$, say g_i. Then ψ is given by polynomials $g_1, \ldots, g_n$ in $k[x_1, \ldots, x_n]$. Now $\varphi \circ \psi = \mathrm{id}_{\mathbb{A}^n}$. This means that

$$f_1(g_1, \ldots, g_n) = x_1, \ldots, f_n(g_1, \ldots, g_n) = x_n$$

This means that (by chain rule)

$$\begin{aligned}\delta_{ij} &= \sum_{k=1}^{n} \frac{\partial f_i(g_1, \ldots, g_n)}{\partial g_k} \cdot \frac{\partial g_k}{\partial x_j}(x_1, \ldots, x_n) \\ &= \sum_{k=1}^{n} \frac{\partial f_i(x_1, \ldots, x_n)}{\partial x_k}(g_1, \ldots, g_n) \cdot \frac{\partial g_k}{\partial x_j}(x_1, \ldots, x_n)\end{aligned}$$

where δ_{ij} is the dirac delta function. Now let

$$J_1(x_1, \ldots, x_n) = \left(\frac{\partial f_i}{\partial x_j}(x_1, \ldots, x_n) \right)_{ij} \quad \text{and} \quad J_2(x_1, \ldots, x_n) = \left(\frac{\partial g_i}{\partial x_j}(x_1, \ldots, x_n) \right)_{ij}$$

be the respective Jacobian matrices of φ and ψ. Then the above equations says that

$$J_1(g_1, \ldots, g_n) J_2(x_1, \ldots, x_n) = \mathrm{id}_{n \times n}$$

Similarly, using that $\psi \circ \varphi = \mathrm{id}_{\mathbb{A}^n}$, we will get that

$$J_2(f_1, \ldots, f_n) J_1(x_1, \ldots, x_n) = \mathrm{id}_{n \times n}$$

This means that $J_1(x_1, \ldots, x_n)$ is an invertible matrix (using that a square matrix with left inverse is invertible). Hence $J(x_1, \ldots, x_n) = \det |\partial f_i / \partial x_j|$ is invertible in $k[x_1, \ldots, x_n]$. i.e. belongs to k^*. □

Exercise I.3.7. Let Y be a variety of dimension $\geqslant 2$, and let $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.

(a) Show that f extends to a regular function on Y.

(b) Show this would be false for $\dim Y = 1$.

See (III, Ex. 3.5) for generalization.

Solution. (a) Let $\dim Y = r$ and $X = Y - \{P\}$, an open subset of Y. Since every variety is covered by quasi-affine varieties, WLOG we can assume that Y is quasi-affine. So $Y = U \cap Z$ where Z is an affine variety of dimension r and U is an open subset of $\mathbb{A}^n$ where $Z \subset Z$. Then the point $P \in Y$ corresponds to a maximal ideal $\mathfrak{m}_P$ of $A(Z)$. And $\mathcal{O}_{Y,P} = A(Z)_{\mathfrak{m}_P}$ which is an integrally closed domain. Now we will use the following result from commutative algebra:

Lemma I.3.1. Let A be a commutative, Noetherian ring which is integrally closed. Then

$$A = \bigcap_{\mathfrak{p}} A_{\mathfrak{p}}$$

where $\mathfrak{p}$ varies over all height 1 prime ideals and intersection is taking place inside $K(A)$.

Let $\mathfrak{p}$ be a height 1 prime ideal in $\mathcal{O}_{Y,P}$. Then it will correspond to a height 1 prime ideal, also denoted by $\mathfrak{p}$, of $A(Z)$ contained in $\mathfrak{m}_P$. So $\mathcal{Z}(\mathfrak{p})$ will define a codimension 1 affine variety in Z containing the point P. In particular, $\mathcal{Z}(\mathfrak{p}) \cap Y \neq \emptyset$. Now because $\dim Z = r \geq 2$, $\dim \mathcal{Z}(\mathfrak{p}) \geq 1$ and therefore $\dim Y \cap \mathcal{Z}(\mathfrak{p}) \geq 1$ (as $Y \cap \mathcal{Z}(\mathfrak{p})$ is an non-empty open subset of $\mathcal{Z}(\mathfrak{p})$ and is therefore dense in it. Also use proposition I.1.10 that $\dim Y = \dim \bar{Y}$), so $\mathcal{Z}(\mathfrak{p})$ will have non-empty intersection with $X = Y - \{P\}$. Now consider

$$f|_{X \cap \mathcal{Z}(\mathfrak{p})}$$

which is a regular function on $X \cap \mathcal{Z}(\mathfrak{p})$. Around any point $Q \in X \cap \mathcal{Z}(\mathfrak{p})$, we can find an open subset $V \subset X \cap \mathcal{Z}(\mathfrak{p})$ such that

$$f|_{X \cap \mathcal{Z}(\mathfrak{p})} = g/h \quad \text{where} g, h \in A(Z)$$

Claim: $h \notin \mathfrak{p}$. Because if it did then $h(Q') = 0$ for all $Q' \in V$ as $V \subset \mathcal{Z}(\mathfrak{p})$. Contradiction!

But then this means that

$$f|_{X \cap \mathcal{Z}(\mathfrak{p})} \in A(Z)_{\mathfrak{p}}$$

This happens for each height 1 prime $\mathfrak{p}$ of $A(Z)$. Now by above lemma

$$\mathcal{O}_{Y,P} = A(Z)_{\mathfrak{m}_P} = \bigcap_{\mathfrak{p} \subset A(Z)_{\mathfrak{m}_P}, \text{height}\, \mathfrak{p} = 1} (A(Z)_{\mathfrak{m}_P})_{\mathfrak{p}} = \bigcap_{\mathfrak{p} \subset \mathfrak{m}_P \text{ in } A(Z), \text{height}\, \mathfrak{p} = 1} A(Z)_{\mathfrak{p}}$$

This means that f is regular at P! Hence f is regular on whole of Y.

(b) When $\dim Y = 1$ then take $Y = \mathbb{A}^1$ and $f(x) = 1/x$ which is defined on $\mathbb{A}^1 - \{0\}$. Then f cannot be extended to the whole of $\mathbb{A}^1$ because if it did then in the neighbourhood U of 0, it is given by ratio of two polynomials $f(x)/g(x)$ where $g(0) \neq 0$. This must also match with $1/x$ on $U - \{0\}$. This means $xf(x) = g(x)$ implying $g(0) = 0$. Contradiction! □

§§I.4. Rational maps

Exercise I.4.1. If f and g are regular functions on open subsets U and V of a variety X, and if $f = g$ on $U \cap V$, show that the function which is f on U and g on V is a regular function on $U \cup V$.

Conclude that if f is a rational function on X, then there is a largest open subset U of X on which f is represented by a regular function. We say that f is defined at the points of U.

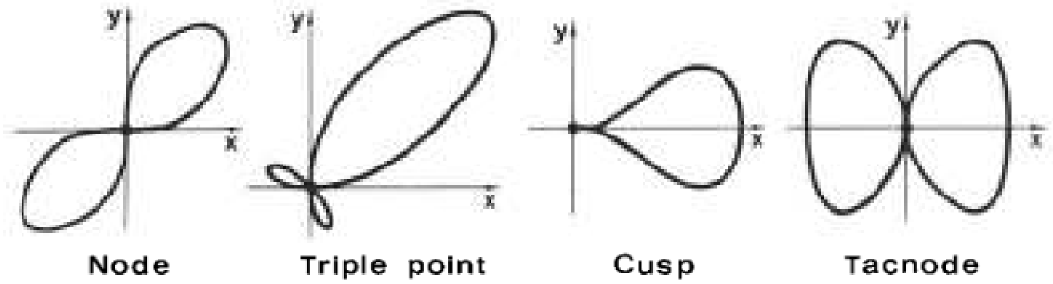


Figure 4. Singularities of plane curves.

Solution. We define a map

$$F : U \cup V \longrightarrow \mathbb{A}^1, \qquad F(P) = \begin{cases} f(P) & \text{if } P \in U \\ g(P) & \text{if } P \in V \end{cases}$$

Then F is a well-defined function since $f = g$ on $U \cap V$. This is a regular function because in neighbourhood of any point $P \in U \cup V$, we can find a neighbourhood where f is given by ratio of two polynomials. The second statement is clear from the first. □

§§I.5. Non-singular curves

Exercise I.5.1. Locate the singular points and sketch the following curves in $\mathbb{A}^2$ (assume char $k \neq 2$). Which is which in Figure 4?

(a) $x^2 = x^4 + y^4$;

(b) $xy = x^6 + y^6$;

(c) $x^3 = y^2 + x^4 + y^4$;

(d) $x^2y + xy^2 = x^4 + y^4$.

Solution. (a) Let $f(x,y) = x^2 - x^4 - y^4$. Then

$$\frac{\partial f}{\partial x}(x,y) = 2x - 4x^3 \quad \text{and} \quad \frac{\partial f}{\partial x}(x,y) = -4y^3$$

Now

$$\frac{\partial f}{\partial x}(x,y) = 0 \quad \Longrightarrow \quad x = 0, 1/\sqrt{2}, -1/\sqrt{2}$$

And

$$\frac{\partial f}{\partial y}(x,y) = 0 \quad \Longrightarrow \quad y = 0$$

Now putting $f(x,0) = x^2 - x^4 = 0$ gives $x = 0, 1, -1$. So $(0,0)$ is the only singular point of this curve. Now, from the equation, we see that this curve is symmetric about both x and y axes. So it must have tacnode. Another way to see this is that to find tangent lines at $(0,0)$, we factorize the lowest order homogeneous term which here is $x^2 = x \cdot x$. So at $(0,0)$, it has two tangent line, both of which are the same $x = 0$ i.e., y-axis. So it has tacnode.

(b) Let $f(x,y) = xy - x^6 - y^6$. Then

$$\frac{\partial f}{\partial x}(x,y) = y - 6x^5 \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = x - 6y^5$$

Equating $\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 0$, we get that

$$y = 6^6 y^{25} \implies y = 0, \zeta_{24}^i 6^{1/4} \quad i = 0, 1, \ldots, 23$$

where ζ_{24} is a primitive 24^{th} root-of-unity in k (which exists since k is algebraically closed). Putting these values in the second equation, we get that

$$x = 0, \zeta_{24}^i 6^{9/4} \quad i = 0, 1, \ldots, 23$$

So (x,y) where x and y are one of those above values is a singular point if it lies on the curve. A quick check gives us that only $(0,0)$ lies on the curve. So $(0,0)$ is the only singular point. To find tangent lines at $(0,0)$, we factorize the lowest order homogeneous term which here is $xy = x \cdot y$. So at $(0,0)$, it has two tangent lines $x = 0$ and $y = 0$ i.e., x-axis and y-axis. So it has node.

(c) Let $f(x,y) = x^3 - y^2 - x^4 - y^4$ then

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 - 4x^3 \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = -2y - 4y^3$$

Equating $\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 0$, we get that

$$x = 0, 3/4 \quad \text{and} \quad y = 0, 1/\sqrt{-2}, -1/\sqrt{-2}$$

So (x,y) where x and y are one of those above values is a singular point if it lies on the curve. A quick check gives us that only $(0,0)$ lies on the curve. So $(0,0)$ is the only singular point. To find tangent lines at $(0,0)$, we factorize the lowest order homogeneous term which here is $y^2 = y \cdot y$. So at $(0,0)$, it has two tangent line, both of which are the same $y = 0$ i.e., the x-axis. So it has node.

(d) Let $f(x,y) = x^2y + xy^2 - x^4 - y^4$ then

$$\frac{\partial f}{\partial x}(x,y) = 2xy + y^2 - 4x^3 \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = 2xy + x^2 - 4y^3$$

Equating $\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 0$, we get that

$$\begin{aligned} 0 = 2xy + x^2 - 4y^3 &= x^2 + y(2x - 4y^2) \\ &= x^2 + y(2x - 4(4x^3 - 2xy)) \quad \text{because of first equation} \\ &= x^2 + 8xy^2 + y(2x - 16x^3) \\ &= x(x + 8(4x^3 - 2xy) + y(2 - 16x^2)) \\ &= x(x + 32x^3 - y(2 - 16x - 16x^2)) \end{aligned}$$

$x = 0$ is clearly a solution which gives $y = 0$ from the equation. Now we seek other solutions. A quick check shows that if $2 - 16x - 16x^2 = 0$. Then $x + 32x^3 = 0$. These two equations have no common solutions. So $2 - 16x - 16x^2 \neq 0$ and $y = (x + 32x^3)/(2 - 16x - 16x^2)$. So $(0,0)$ is the only singular point. To find tangent lines at $(0,0)$, we factorize the lowest order homogeneous term which here is $x^2y + xy^2 = x \cdot y \cdot (x + y)$. So it has three tangent lines at the origin i.e., it has triple point. □

Definition I.5.1. Let $Y \subseteq \mathbb{A}^2$ be a curve defined by the equation

$$f(x,y) = 0.$$

Let $P = (a,b)$ be a point of $\mathbb{A}^2$. Make a linear change of coordinates so that P becomes the point $(0,0)$. Then write f as a sum

$$f = f_0 + f_1 + \ldots + f_d,$$

where f_i is a homogeneous polynomial of degree i in x and y. Then we define the **multiplicity** of P on Y, denoted $\mu_P(Y)$, to be the least r such that $f_r \neq 0$.

The linear factors of f are called the **tangent directions** at P.

Exercise I.5.2. (a) Show that $\mu_P(Y) = 1 \iff P$ is a nonsingular point of Y.

(b) Find the multiplicity of each of the singular points in (Ex. 5.1) above.

Solution. (a) ($\Longrightarrow$) Since $(0,0)$ lies on the curve, $f_0 = 0$. Let $f_1 = ax + by$. Then $\mu_P(Y) = 1$ implies that either $a \neq 0$ or $b \neq 0$. But this means that

$$\text{either} \qquad \frac{\partial f}{\partial x}(0,0) = a \neq 0 \qquad \text{or} \qquad \frac{\partial f}{\partial y}(0,0) = b \neq 0.$$

So $P = (0,0)$ is a non-singular point.

($\Longleftarrow$) We reverse the above arguments. Since $P = (0,0)$ lies on the curve, $f_0 = 0$. Let $f_1 = ax + by$. Since $(0,0)$ is a non-singular point, either $\frac{\partial f}{\partial x}(0,0) = a \neq 0$ or $\frac{\partial f}{\partial y}(0,0) = b \neq 0$. So $\mu_P(Y) = 1$.

(b) In Exercise 5.3, all curves had only one singular point, that too at the origin $P = (0,0)$. It is clear from the equations that

(a) $x^2 = x^4 + y^4, \qquad \mu_P(Y) = 2;$

(b) $xy = x^6 + y^6, \qquad \mu_P(Y) = 2;$

(c) $x^3 = y^2 + x^4 + y^4, \qquad \mu_P(Y) = 2;$

(d) $x^2y + xy^2 = x^4 + y^4, \qquad \mu_P(Y) = 3.$

□

Exercise I.5.3. (Analytically Isomorphic Singularities)

(a) If $P \in Y$ and $Q \in Z$ are analytically isomorphic plane curve singularities, show that the multiplicities $\mu_P(Y)$ and $\mu_Q(Z)$ are the same (Ex. 5.3).

(b) Generalize the example in the text (5.6.3) to show that if $f = f_r + f_{r+1} + \ldots \in k[[x,y]]$, and if the leading form f_r of f factors as $f_r = g_s h_t$, where g_s, h_t are homogeneous of degrees s and t respectively, and have no common linear factor, then there are formal power series

$$g = g_s + g_{s+1} + \ldots$$
$$h = h_t + h_{t+1} + \ldots$$

in $k[[x,y]]$ such that $f = gh$.

(c) Let Y be defined by the equation

$$f(x,y) = 0 \qquad \textbf{in } \mathbb{A}^2$$

and let $P = (0,0)$ be a point of multiplicity r on Y, so that when f is expanded as a polynomial in x and y, we have $f = f_r+$ higher order terms. We say that P is an **ordinary r-fold point** if f_r is a product of r distinct linear factors.

Show that any two ordinary double points are analytically isomorphic.

Ditto for ordinary triple points.

But show that there is a one-parameter family of mutually non-isomorphic ordinary 4-fold points.

Solution. (a) Suppose Y and Z are given by

$$f(x,y) = f_r + \ldots + f_d \quad \text{and} \quad g(x,y) = g_s + \ldots + g_e$$

where f_r and g_s are the lowest degree homogeneous term of f and g respectively. Now

$$\widehat{\mathcal{O}}_{Y,P} \cong k[\![x,y]\!]/(f(x,y)) \quad \text{and} \quad \widehat{\mathcal{O}}_{Z,Q} \cong k[\![x,y]\!]/(g(x,y))$$

These both are local rings with maximal ideal $\mathfrak{m}_P = (x,y)/(f(x,y))$ and $\mathfrak{m}_Q = (x,y)/(g(x,y))$. Since these two are isomorphic, there is an automorphism φ of $k[\![x,y]\!]$ which maps (x,y) to itself and the ideal $(f(x,y))$ to the ideal $(g(x,y))$. In particular, φ is continuous with respect to the $\mathfrak{m}$-adic topology of $R = k[\![x,y]\!]$. Since $k[x,y]$ is dense in this $\mathfrak{m}$-adic topology, φ is determined by where it sends $k[x,y]$ which is determined by where it sends x and y. Moreover, if we are given $a, b \in \mathfrak{m}$, there is a unique continuous k-algebra homomorphism $\psi : R \longrightarrow R$ such that $\psi(x) = a$ and $\psi(y) = b$. So the only question is what conditions on a and b guarantee that this ψ is an automorphism.

Claim: ψ is an automorphism $\iff$ images of a and b in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent.

Proof. ($\Longrightarrow$) ψ induces a vector space isomorphism of $\mathfrak{m}/\mathfrak{m}^2$. Since x and y are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$, so must be $a = \psi(x)$ and $b = \psi(y)$.

($\Longleftarrow$) Suppose images of a and b in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent. This just means that the linear homogeneous parts of a and b are linearly independent. First we prove that ψ

is surjective: Let $a = a_1 + a_2 + \ldots$ and $b = b_1 + b_2 + \ldots$ and suppose we are given $q = q_0 + q_1 + q_2 + \ldots \in R$. Let $p(x, y) \in R$ be

$$p(x, y) = p_{0,0} + p_{2,0}x^2 + p_{1,1}xy + p_{0,2}y^2 + \ldots$$

Find coefficients $p_{i,j}$ such that $\psi(p) = p(a, b) = q$. This can be done inductively. For example, $p_{0,0} = q_0$. This proves that ψ is surjective. Now ψ is a surjective endomorphism of a Noetherian ring. So it must be injective (otherwise we will have an infinite strictly increasing chain of ideals $\ker\psi \subseteq \ker\psi^2 \subseteq \ldots$). □

Back to our original question: φ was an automorphism of $R = k[\![x, y]\!]$. So it is given by elements $a, b \in (x, y)$ with linearly independent linear terms. Since ψ also takes the ideal $(f(x, y))$ to the ideal $(g(x, y))$, we have that $f(a, b) = g(a, b)u$ where u is a unit in $k[\![x, y]\!]$. This just means that leading degrees r and s of f and g must be the same.

(b)

(c) Suppose $f(x, y) = (\alpha x + \beta y)(\alpha' x + \beta' y) + h.o.t$ where $\alpha\beta' - \alpha'\beta \neq 0$. Now we have

$$\widehat{\mathcal{O}}_{P,Y} \cong k[\![x, y]\!]/(f(x, y))$$

As we did in Example I.5.6.3, we can write $f = gh$ where

$$g = (\alpha x + \beta y) + h.o.t. \quad \textbf{and} \quad g = (\alpha' x + \beta' y) + h.o.t. \quad \text{in} \quad k[\![x, y]\!]$$

(Note that as $\alpha\beta' - \alpha'\beta \neq 0$, $(\alpha x + \beta y)$ and $(\alpha' x + \beta' y)$ generates the maximal ideal of $k[\![x, y]\!]$). Because $\alpha\beta' - \alpha'\beta \neq 0$, g and h begin with linearly independent linear terms. Hence there is an automorphism of $k[\![x, y]\!]$ sending g to x and h to y. So

$$\widehat{\mathcal{O}}_{P,Y} \cong k[\![x, y]\!]/(xy)$$

So all double points are analytically isomorphic.

Now we come to triple points. Suppose

$$f(x, y) = (\alpha x + \beta y)(\alpha' x + \beta' y)(\alpha'' x + \beta'' y) + h.o.t$$

where the linear terms are linearly independent. We can write

$$\alpha'' x + \beta'' y = a(\alpha x + \beta y) + b(\alpha' x + \beta' y)$$

Again, as we did in Example 1.5.6.3, we can write $f = gh(ag + bh)$ where

$$g = (\alpha x + \beta y) + h.o.t \quad \textbf{and} \quad g = (\alpha' x + \beta' y) + h.o.t \quad \text{in} \quad k[\![x, y]\!]$$

Again, this will use that as $\alpha\beta' - \alpha'\beta \neq 0$, $(\alpha x + \beta y)$ and $(\alpha' x + \beta' y)$ generates the maximal ideal of $k[\![x, y]\!]$. Further making linear change of coordinates, can make $f = gh(g + h)$. Because $\alpha\beta' - \alpha'\beta \neq 0$, g and h begin with linearly independent linear terms. Hence there is an automorphism of $k[\![x, y]\!]$ sending g to x and h to y. So

$$\widehat{\mathcal{O}}_{P,Y} \cong k[\![x, y]\!]/(xy(x + y))$$

So all triple points are analytically isomorphic.

Now we come to ordinary 4-fold points. Doing the above process again, we will get that

$$\widehat{\mathcal{O}}_{P,Y} \cong k[\![x,y]\!]/(xy(x+y)(x+ty))$$

where $t \neq 0,1$ is a parameter. So we have a one-parameter family of non-isomorphic ordinary 4-fold points. □

§II. Schemes

Exercise II.0.1. Assume all the schemes below are noetherian.

(a) Closed immersions and open immersions are seperated.

(b) Composition of separated morphisms is separated.

(c) Seperatedness is preserved by base change.

(d) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms such that $g \circ f$ is separated then f is separated.

(e) Seperatedness is local on the base. i.e., A morphism $f : X \longrightarrow Y$ is seperated iff Y is covered by open subsets V_i such that $f^{-1}(V_i) \longrightarrow V_i$ is seperated for each i.

Proof. (a) Cover $f : X \longrightarrow Y$ be a closed immersion. Cover Y by affine open subsets V_i. Then $f^{-1}(V_i)$ is affine scheme equal to $\operatorname{Spec} B/\mathfrak{b}$, $\mathfrak{b} \subset B$ where $V_i = \operatorname{Spec} B$. Now the result follows from part (e) (which will be proved independently) and the fact that a morphism of affine schemes is separated.

Let $f : U \hookrightarrow X$ be an open immersion. Here we are assuming that $U \subset X$ an open subset. We will use valuative criterion of separatedness to prove that f is separated: Given a commutative diagram

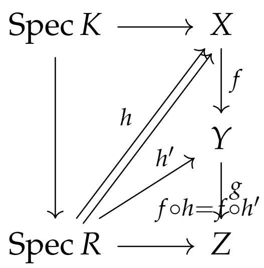
$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{g'} & U \\ \downarrow & \overset{h_1}{\underset{h_2}{\nearrow\!\!\!\nearrow}} & \big\downarrow f \\ \operatorname{Spec} R & \xrightarrow{g} & X \end{array}$$

Because f is an inclusion and $f \circ h_1 = g = f \circ h_2$, we have that $h_1 = h_2$ as map of topological spaces. Also the maps of sheaves $\mathcal{O}_U \longrightarrow (h_1)_*(\mathcal{O}_{\operatorname{Spec} R})$ and $\mathcal{O}_U \longrightarrow (h_2)_*(\mathcal{O}_{\operatorname{Spec} R})$ are the same because they both equal to the restriction of map of sheaves $Z \longrightarrow g_*(\mathcal{O}_{\operatorname{Spec} R})$ to the open subset U.

(b) Suppose we are given two separated morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$. We want to show that $g \circ f$ is separated. We will use valuative criterion of sepataredness for this. Suppose we are given a diagram (Figure 1).

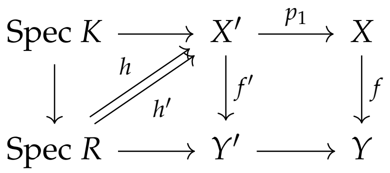
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ & & \big\downarrow f \\ \big\downarrow & \overset{h}{\underset{h'}{\nearrow\!\!\!\nearrow}} & Y \\ & & \big\downarrow g \\ \operatorname{Spec} R & \longrightarrow & Z \end{array} \qquad \begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ & & \big\downarrow f \\ \big\downarrow & & Y \\ & \overset{f\circ h}{\underset{f\circ h'}{\nearrow\!\!\!\nearrow}} & \big\downarrow g \\ \operatorname{Spec} R & \longrightarrow & Z \end{array}$$

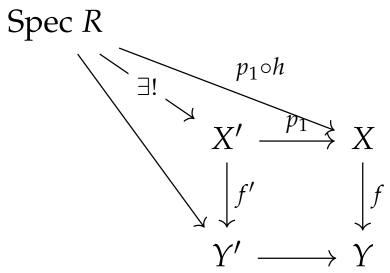
Then composing with f, we obtain a diagram as in Figure 2. Since g is separated, we have that $f \circ h = f \circ h'$. So now we obtain the following commutative diagram:



Now since f is separated, $h = h'$.

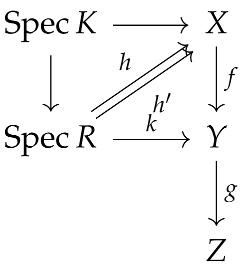
(c) Let $f : X \longrightarrow Y$ be a seperated morphism and $Y' \longrightarrow Y$ be any morphism. We must show h and h' as shown in figure are the same maps.

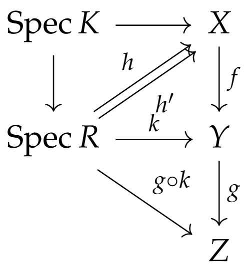




By the valuative criterion of separatedness, $p_1 \circ h = p_1 \circ h'$. By the universal property of fibered products, $h = h'$.

(d) Again we use the valuative criterion to prove that f is separated. Suppose we are given a commutative diagram as in figure 1. Then composing g with $k : \operatorname{Spec} R \longrightarrow Y$, we obtain a commutative diagram as in figure 2:





Since $g \circ f$ is separated, we have that $h = h'$.

(e) $(\implies)$ By part (c), $f^{-1}(V_i) \longrightarrow V_i$ is seperated since it is base change by inclusion $V_i \hookrightarrow X$.

$(\impliedby)$ To check that $\Delta : X \longrightarrow X \times_Y X$ an closed immersion, it suffices to check it on an open cover. If $g : X \times_Y X \longrightarrow Y$ is the natural morphism, then open cover $\{V_i\}$ of Y gives us an open cover

$$g^{-1}(V_i) = f^{-1}(V_i) \times_{V_i} f^{-1}(V_i) \quad \text{of} \quad X \times_Y X.$$

Now $f^{-1}(V_i) \longrightarrow V_i$ separated implies that the morphism $f^{-1}(V_i) \longrightarrow f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$ is a closed immersion. This happens for each i. So Δ is also a closed immersion. □

Exercise II.0.2. (Exercise II.3.13)

(a) A closed immersion is a morphism of finite type.

(b) A composition of two morphisms of finite type is of finite type.

(c) Morphisms of finite type are stable under base extension.

(d) A closed immersion is stable under base change.

Solution. (a) Let $f : X \longrightarrow Y$ be a closed immersion. Cover Y by affine opens V_i =Spec A_i. Then $f^{-1}(V_i)$ =Spec A_i/I for some ideal $I \subseteq A_i$. Clearly, A_i/I is a finitely generated A_i-module. In particular, it is a finitely generated A_i-algebra.

(b) Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms of finite type. Let $h = g \circ f$ and $V = \operatorname{Spec} C \subseteq Z$ be an affine open. Since g is of finite type, by Exercise II.3.3, we have that

$$g^{-1}(V) = \bigcup_{i=1}^{n} \operatorname{Spec} B_i$$

such that B_i is finitely generated C-algebra. Now again using that f is of finite type,

$$f^{-1}(B_i) = \bigcup_{j=1}^{m_i} \operatorname{Spec} A_{ij}$$

where A_{ij} is fintely generated B_i-algebra. Hence

$$h^{-1}(V) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \operatorname{Spec} A_{ij}$$

where A_{ij} is finitely generated C-algebra. So h is a finite type morphism.

(c) Let $f : X \longrightarrow Y$ be a finite morphism and $f' : X' \longrightarrow Y'$ be a base extension of f

$$\begin{array}{ccc} X' = X \times_Y Y' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\quad g \quad} & Y \end{array}$$

Let $U = \operatorname{Spec} B$ be an affine open in Y such that $g^{-1}(U) \neq \emptyset$ and let $V = \operatorname{Spec} A' \subseteq g^{-1}(U)$. Since f is of finite type, by exercise II.3.3, we can write $f^{-1}(U) = \cup_{i=1}^{n} \operatorname{Spec} A_i$ as finite union where A_i's are finitely generated B-algebras. Then

$$\begin{aligned} f'^{-1}(V) &= V \times_Y X \\ &= V \times_{\operatorname{Spec} B} f^{-1}(U) \\ &= \bigcup_{i=1}^{n} \operatorname{Spec}(A_i \otimes_B A'). \end{aligned}$$

If $\{b_1, \ldots b_r\}$ is finite generating set of A_i as an B-algebra then $\{b_1 \otimes 1, \ldots b_r \otimes 1\}$ is a finite generating set of $(A_i \otimes_B A')$ as an A'-algebra.

Now cover Y with open affines $\{U_i\}_i$. Then we can cover $g^{-1}(U_i)$ by open affines $V_{ij} = \operatorname{Spec} B'_{ij}$. Then $f'^{-1}(V_{ij})$ can be covered by finitely many open affines $\operatorname{Spec} A'_{ijk}$ where A'_{ijk}'s are finitely generated B'_{ij}-algebras. Hence the morphism f' is of finite type.

(d) This corresponds to the fact that tensor product is right exaxt. Let $f : X \longrightarrow Y$ be a closed immersion and $f' : X' \longrightarrow Y'$ be its base change by a morphism $g : Y' \longrightarrow Y$. Cover Y by affine opens $\{U_i = \operatorname{Spec} A_i\}$ and cover $g^{-1}(U_i)$ by open affines $\{V_{ij} = \operatorname{Spec} C_{ij}\}$ where C_{ij} are A_i-modules. Since f is a closed immersion, $f^{-1}(U_i) = \operatorname{Spec} B_i$ where $A_i \longrightarrow B_i$ is a surjective homomorphism. Then

$$f'^{-1}(V_{ij}) = V_{ij} \times_Y X = V_{ij} \times_{U_i} f^{-1}(U_i) = \operatorname{Spec}(C_{ij} \otimes_{A_i} B_i)$$

Since tensor product is right exaxt, $C_{ij} \longrightarrow C_{ij} \otimes_{A_i} B_i$ is surjective. Hence $f'^{-1}(V_{ij}) \longrightarrow V_{ij}$ is a closed immersion hence f' is a closed immersion. □

Exercise II.0.3. Assume all the schemes below are noetherian.

(a) Closed immersions are proper.

(b) A composition of proper morphisms is proper.

(c) Proper morphisms are stable under base change.

(d) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are morphisms such that $g \circ f$ is proper and g is separated. Then f is proper.

(e) Properness is a local property on the base.

Solution. (a) Let $f : X \longrightarrow Y$ be a closed immersion.

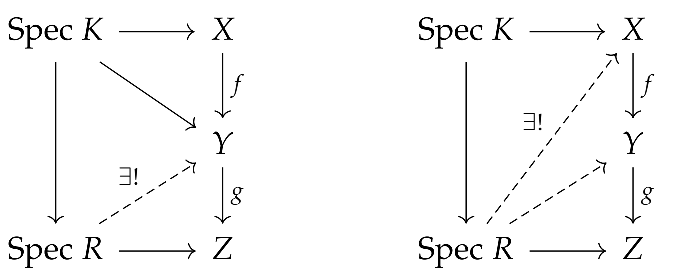
(a) By Exercise II.3.13(c), closed immersions are stable under base extension. Closed immersions are ofcourse closed. So f is universally closed.

(b) By Exercise II.3.11(a), f is a finite type morphism.

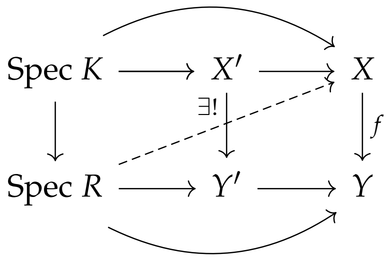
(c) Also we saw that closed immersions are separated.

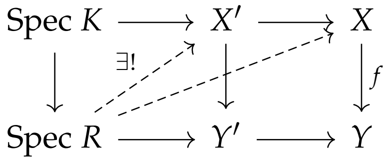
So f is proper.

(b) Let f and g are proper morphisms. By Exercise II.3.13(b), $g \circ f$ is of finite type. So we can use the valuative criterion of properness to check properness of $g \circ f$.



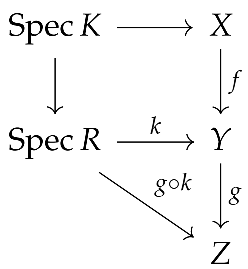
(c) Let $f : X \longrightarrow Y$ be a proper morphism and $f' : X' \longrightarrow Y'$ be its base change to $Y' \longrightarrow Y$. By Exercise II.3.13(c), f' is of finite type. Since Y' is noetherian, we can use valuative criterion to check properness of f'. The following diagram is self explainatory:

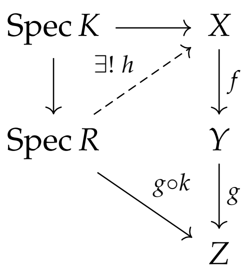




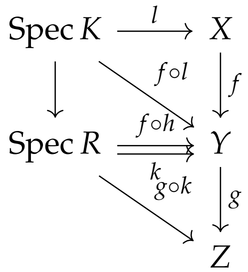
For the second diagram, we used the universal property of fibered products.

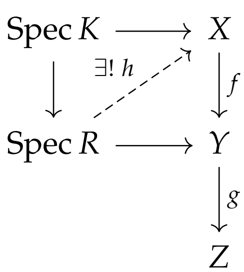
(d) We will use valuative criterion of properness to prove this. Suppose that we are given a commutative diagram as in figure 1 then using that $g \circ f$ is proper, we get a commutative diagram as in figure 2





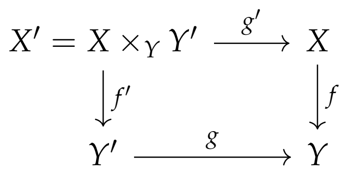
Now we have the following diagram (figure 3) obtained by composing morphisms





Since g is separated, $f \circ h = k$ i.e., the lower triangle of figure 4 commutes. By valuative criterion of properness, f is proper.

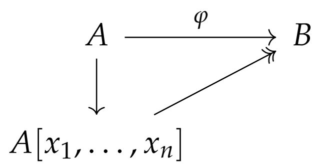
(e) Suppose that Y is covered by open subsets $\{V_i\}$ such that $f^{-1}(V_i) \longrightarrow V_i$ is proper for each i. Since separatedness is a local property on base, f is separated. Clearly, this also implies that f is of finite type. Suppose we given a morphism $g : Y' \longrightarrow Y$. Then we obtain the base extension



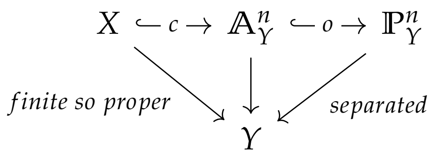
Now we have that $f^{-1}(V_i) \times_Y Y' = f^{-1}(V_i) \times_{V_i} g^{-1}(V_i)$. Also $X \times_Y Y'$ is covered by $\{f^{-1}(V_i) \times_Y Y'\}$ and f is just the glueing of restriction morphisms $f^{-1}(V_i) \times_Y Y' \longrightarrow Y'$ which actually are the morphisms $f^{-1}(V_i) \times_{V_i} g^{-1}(V_i) \longrightarrow g^{-1}(V_i)$ which are the base extension of $f^{-1}(V_i) \longrightarrow V_i$ by $g^{-1}(V_i) \longrightarrow V_i$. So they are closed morphisms. Since checking that whether a morphism is closed or not can be done locally on the base and $\{g^{-1}(V_i)\}$ cover Y', f' is a closed morphism. Since f' was an arbitrary base extension, f is a universally closed. □

Exercise II.0.4. Finite maps are projective.

Solution. We prove this for map of affine schemes. Suppose $\varphi : A \longrightarrow B$ be a map of rings such that B is a finitely generated module over A. Then for some $n \in \mathbb{N}$, we have



Suppose $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$. Then we get



By above part (d), $X \longrightarrow \mathbb{P}^n_Y$ is proper. In particular, it is closed. Since it is composition of immersions, it is still an immersion so it is a closed immersion. Therefore f is projective. □