INDIAN INSTITUTE OF SCIENCE

भारतीय विज्ञान संस्थान

MA220: Representation Theory of Finite Groups

Topic: Pontryagin duality

Professor: Dr. Radhika Ganapathy

Notes By: Ajay Prajapati

SR No: 21523

Autumn 2022

Contents

§1. Introduction

Let G be a locally compact and Hausdorff abelian group. Examples of such groups are finite groups (with discrete topology), $S^1 := \mathbb{R}/\mathbb{Z}$ (the circle group which can also be thought of as subgroup of $\mathbb{C}^*$), $\mathbb{R}$, $\mathbb{Q}_p$, any finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{Q}_p$, etc.

Definition 1.1. A *character* of G is a continuous homomorphism $\chi : G \longrightarrow S^1$.

Let $\widehat{G}$ denote the set of characters of G. It is an abelian group under pointwise multiplication.

Example 1.1. Let $G = \mathbb{R}$. Then for any $x \in \mathbb{R}$, the function

$$\mathbb{R} \longrightarrow S^1, \qquad y \longmapsto e^{2\pi i x y}$$

is a character of $\mathbb{R}$. Infact, these are all the characters of $\mathbb{R}$ (see [Con19], chapter VII, theorem 9.11). The main idea is to observe that any $\chi \in \widehat{\mathbb{R}}$ should be differentiable. Then use multiplicativity of χ to set up a differential equation solving which gets us the result. So $\widehat{\mathbb{R}} \cong \mathbb{R}$.

Let $G = \mathbb{R}/\mathbb{Z}$ then for any $m \in \mathbb{Z}$, the function $y \longmapsto e^{2\pi i m y}$ is a character of G. Infact, these are all the characters of G (This follows from the above result). So $\widehat{G} \cong \mathbb{Z}$.

For $G = \mathbb{Q}$ (or $\mathbb{Q}_p$), see the brilliant article of Keith Conrad [Keib]. It turns out that $\widehat{\mathbb{Q}} \cong \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$, the group of rational adèles modulo the rational numbers (which are embedded naturally into the adèle ring via the diagonal map). Similar to $\mathbb{R}$, all characters of $\mathbb{Q}_p$ are of the form $y \longmapsto e^{2\pi i x y}$ for some $x \in \mathbb{Q}_p$ (Note that if $t \in \mathbb{Q}_p$ then $e^{2\pi i t} = e^{2\pi i a/p^N}$ for $a, N \in \mathbb{Z}_{\geq 0}$ such that $t - a/p^N \in \mathbb{Z}_p$).

In section 2, we will see that when G is a finite abelian group then G and $\widehat{G}$ are isomorphic, but non-canonically, and G and $\widehat{\widehat{G}}$ are canonically isomorphic (given by the evaluation map).

When G is infinite we do not get that G and $\widehat{G}$ are isomorphic (see example 1.1) but it will still be true that G and $\widehat{\widehat{G}}$ are naturally isomorphic. In section 4, first we will topologize $\widehat{G}$ and prove that $\widehat{G}$ itself is locally compact. Then the Pontryagin duality theorem states that:

Theorem 1.2. [Pontryagin duality] The mapping

$$\mathrm{ev}_G : G \longrightarrow \widehat{\widehat{G}}, \qquad g \longmapsto (\mathrm{ev}_G(g) : \chi \longmapsto \chi(g))$$

is an isomorphism of topological groups. Hence G and $\widehat{G}$ are mutually dual.

The principal technical tool for establishing this theorem is the Fourier inversion formula which we will state in 3. The main reference for this note is [RV13], chapter 3.

§2. Pontryagin duality for finite abelian groups

In this section we see that when G is a finite abelian group then $G \cong \widehat{\widehat{G}}$ naturally. First we explicitly calculate $\widehat{G}$ for a finite cyclic group and then use the structure theorem for finite abelian groups to calculate $\widehat{G}$ for arbitrary G.

Let $G = \mathbb{Z}/m\mathbb{Z}$ be finite cyclic (with the discrete topology). Then every homomorphism $\chi : G \longrightarrow S^1$ is continuous and is determined by $\chi(\overline{1})$. But also we have $\chi(\overline{1})^m = \chi(m\overline{1}) = \chi(\overline{0}) = 1$. So $\chi(\overline{1})$ is a m^{th} root of unity. Fix a primitive m^{th} root of unity ζ_m. Then

$$\widehat{G} = \{\chi_a : \text{for } a \in \mathbb{Z}/m\mathbb{Z} \text{ such that } \chi_a(\overline{1}) = \zeta_m^a\}$$

It is easy to see that $\chi_a\chi_b = \chi_{a+b}$ and that $\chi_{\overline{1}}$ generates the group $\widehat{G}$. So $\widehat{G} \cong \mathbb{Z}/m\mathbb{Z} \cong G$. Also it is easy to see that if $a \in \mathbb{Z}/m\mathbb{Z}$ is non-trivial then $\chi_{\overline{1}}(a) \neq 1$. So if there is $a \in \mathbb{Z}/m\mathbb{Z}$ such that $\chi(a) = 1$ for all characters χ then $a = \overline{0}$. This will be the key observation for proving the Pontraygin duality for finite groups.

Now let G is an arbitrary finite group. Then by the structure theorem of finite abelian groups it can be (uniquely) broken into product of cyclic groups, say

$$G \cong \frac{\mathbb{Z}}{m_1\mathbb{Z}} \times \frac{\mathbb{Z}}{m_2\mathbb{Z}} \times \ldots \times \frac{\mathbb{Z}}{m_t\mathbb{Z}} \qquad \text{where } m_1|m_2|\ldots|m_t$$

Now for $1 \leq i \leq t$, let χ_i be a character on $\mathbb{Z}/m_i\mathbb{Z}$. Then $\chi = \chi_1\chi_2\ldots\chi_t$ is a character on G defined as: Any $g \in G$ can be written as $g = (g_1, \ldots, g_t)$ then $\chi(g) = \chi_1(g_1)\chi_2(g_2)\ldots\chi_t(g_t)$. Morover, every character χ on G can be decomposed as above: We have the inclusion $\mathbb{Z}/m_i\mathbb{Z} \hookrightarrow G$ which gives a character χ_i on $\mathbb{Z}/m_i\mathbb{Z}$. Now for $g = (g_1, \ldots, g_t)$ in G,

$$\begin{aligned}
\chi(g) &= \chi((g_1, \ldots, g_t)) \\
&= \chi((g_1, 0, \ldots, 0) + (0, g_2, 0, \ldots, 0) + \ldots + (0, \ldots, 0, g_t)) \\
&= \chi((g_1, 0, \ldots, 0))\chi((0, g_2, 0, \ldots, 0)) \ldots \chi((0, \ldots, 0, g_t)) \\
&= \chi_1(g_1)\chi_2(g_2) \ldots \chi_t(g_t)
\end{aligned}$$

So characters on G are in one-to-one correspondence with product of characters on its cyclic factors. Also it is easy to see that products $\chi_1\chi_2\ldots\chi_t$ are in one-to-one correspondence with ordered pairs $(\chi_1, \chi_2, \ldots, \chi_t)$ (two distinct ordered pairs gives two distinct products). So

$$\widehat{G} = \widehat{\frac{\mathbb{Z}}{m_1\mathbb{Z}}} \times \widehat{\frac{\mathbb{Z}}{m_2\mathbb{Z}}} \times \ldots \times \widehat{\frac{\mathbb{Z}}{m_t\mathbb{Z}}} \cong \frac{\mathbb{Z}}{m_1\mathbb{Z}} \times \frac{\mathbb{Z}}{m_2\mathbb{Z}} \times \ldots \times \frac{\mathbb{Z}}{m_t\mathbb{Z}} \cong G.$$

In particular, $|G| = |\widehat{G}| = |\widehat{\widehat{G}}|$. Now we prove that the evaluation map

$$\mathrm{ev}_G : G \longrightarrow \widehat{\widehat{G}}, \qquad g \longmapsto (\mathrm{ev}_G(g) : \chi \longmapsto \chi(g))$$

is an isomorphism. It is clearly a group homomorphism. It is sufficient to prove injectivity. Suppose $g \in G$ is such that $\chi(g) = 1$ for all $\chi \in \widehat{G}$. Then writing $g = (g_1, \ldots, g_t)$, we get that for all i, $\chi_i(g_i) = 1$ for all characters χ_i on $\mathbb{Z}/m_i\mathbb{Z}$. This means that $g_i = 0$ for all i which implies $g = 0$ in G.

§3. Fourier Transform and the Fourier Inversion Formula

Now let G denotes a locally compact abelian group with bi-invariant Haar measure dx and character group $\widehat{G}$. First we will define a specific class of functions in $L^\infty(G)$ which will be helpful in stating the Fourier inversion formula.

Definition 3.1. A Haar measurable function $\varphi : G \to \mathbb{C}$ in $L^\infty(G)$ is said to be of **positive type** if for any $f \in C_c(G)$ the following inequality holds:

$$\iint \varphi(s^{-1}t) f(s) ds \overline{f(t)} dt \geq 0.$$

Let $V(G)$ denote the $\mathbb{C}$-span of continuous functions of positive type.

Definition 3.2. Let $f \in L^1(G)$. Then we define $\widehat{f} : \widehat{G} \longrightarrow \mathbb{C}$, the *Fourier transform* of f, by

$$\widehat{f}(\chi) = \int_G f(y)\overline{\chi}(y) dy$$

for $\chi \in \widehat{G}$. (This formula makes sense, since for all $y \in G$, $|\chi(y)| = 1$ and therefore $|\widehat{f}(\chi)| \leq ||f||_1 < \infty$ for all $\chi \in \widehat{G}$. In particular, $\widehat{f} \in L^\infty(\widehat{G})$.)

Remark 3.3. When $G = \mathbb{R}$ then $\widehat{G} \cong \mathbb{R}$ (see example 1.1) and we can identify every $t \in \mathbb{R}$ with the character $s \longmapsto e^{ist}$. In this case the formula above reduces to

$$\widehat{f}(t) = \int_{\mathbb{R}} f(s) e^{-ist} ds$$

which is the ordinary Fourier transform of a function defined on $\mathbb{R}$. The point is that despite appearances, this should in fact be regarded as a function on $\widehat{\mathbb{R}}$.

Theorem 3.4. [Fourier Inversion Formula] Let $V^1(G) = L^1(G) \cap V(G)$. There exists a Haar measure $d\chi$ (called the *dual* of the measure dx) on $\widehat{G}$ such that for all $f \in V^1(G)$,

$$f(y) = \int_{\widehat{G}} \widehat{f}(\chi)\chi(y) d\chi.$$

Proof. The proof is long and uses a lot of functional analysis. See [RV13], section 3.3. □

We need to also define Fourier transform of a character measure and see a result about it which will be used crucially in our proof of the Pontryagin duality. Let G and $\widehat{G}$ as above and $\widehat{\mu}$ be a Radon measure on $\widehat{G}$ such that $\widehat{\mu}(\widehat{G})$ is finite.

Definition 3.5. The *Fourier transform of the measure* $\widehat{\mu}$ is a function $T_{\widehat{\mu}} : G \longrightarrow \mathbb{C}$ defined as

$$T_{\widehat{\mu}}(y) = \int_{\widehat{G}} \chi(y) d\widehat{\mu}(\chi)$$

Because $\widehat{\mu}(\widehat{G})$ is finite, one deduces at once that this transform is both continuous and bounded by $\widehat{\mu}(\widehat{G})$ on G. Now we establish a result which will be useful later.

Proposition 3.6. If for $T_{\widehat{\mu}}(y) = 0$ every $y \in G$, then $\widehat{\mu} = 0$.

Proof. **(Outline)** For all $f \in L^1(G)$, $f(y)\overline{\chi}(y)$ is measurable on $G \times \widehat{G}$ and it is easy to check that the conditions of Fubini's theorem hold. So

$$\int \overline{\widehat{f}(\chi)} d\widehat{\mu}(\chi) = \int \overline{f(y)} T_{\widehat{\mu}}(y) dy = 0.$$

But the ring $\{\widehat{f} : f \in L^1(G)\} \subset L^\infty(\widehat{G})$ is dense in $C_0(\widehat{G})$ ([RV13], Prop 3.15). Hence

$$\int g(\chi) d\widehat{\mu}(\chi) = 0$$

for all $g \in C_c(\widehat{G}) \subset C_0(\widehat{G})$. The result then follows at once by the elementary correspondence between Radon measures and integrals ([Rud70], theorem 2.14). □

§4. Pontryagin duality for locally compact abelian groups

First we topologize $\widehat{G}$ with the subspace topology as a subset of the space $C(G, S^1)$ of continuous functions $G \longrightarrow S^1$ with the compact-open topology. That means that the basic open sets around the trivial character $\mathbb{1}$ in $\widehat{G}$ are

$$W(K, V) := \{\chi \in \widehat{G} : \chi(K) \subset V\}$$

for compact K in G and $V \subset S^1$ open. The compact-open topology on $C(G, S^1)$ is Hausdorff, so the topology on $\widehat{G}$ is Hausdorff. With the above topology $\widehat{G}$ is a topological group and it is a closed subset of $C(G, S^1)$ (intuitively, a limit of homomorphisms is a homomorphism).

Notation:

- For $g \in G$, $U \subset G$ will be called a neighbourhood of g if $g \in int(U)$.
- Let $\varphi : \mathbb{R} \longrightarrow S^1, t \longmapsto e^{2\pi i t}$. For $0 < \epsilon \leq 1$, define $N(\epsilon) = \varphi((-\epsilon/3, \epsilon/3))$.
- For $m \in \mathbb{Z}_{\geq 1}$ and $X \subset G$, define $X^{(m)} := \{\prod_{j=1}^{n} x_j : x_j \in X, j = 1, \ldots, n\}$.

Now we establish a technical lemma which will help us to check the continuity of a character by a simple criterion which will simplify the proof of local-compactness of $\widehat{G}$.

Lemma 4.1. Let $m \in \mathbb{Z}_{\geq 1}$ and suppose that $x \in \mathbb{C}$ is such that $x, x^2, \ldots, x^m$ lie in $N(1)$. Then $x \in N(1/m)$. Consequently, if $U \subset G$ containing the identity and $\chi : G \longrightarrow S^1$ is a group homomorphism (not necessarily cts) such that $\chi\left(U^{(m)}\right) \subseteq N(1)$, then $\chi(U) \subseteq N(1/m)$.

Proof. **(Outline)** Let $r \in \mathbb{Z}_{\geq 1}$ such that $x^r \in N(1)$. Then there is $0 \leq q < r$ such that $x \in N(1/r)\varphi(q/r)$. It is easy to see that

$$N\left(\frac{1}{r}\right) \cap N\left(\frac{1}{r+1}\right)\varphi\left(\frac{q}{r+1}\right) \neq \varnothing \implies q = 0.$$

Now use induction to prove the first statement. The second statement follows from it. □

Theorem 4.2. 1. A group homomorphism $\chi : G \longrightarrow S^1$ is continuous $\iff$ $\chi^{-1}(N(1))$ is a neighborhood of the identity in G.

2. The family $\{W(K, N(1))\}_K$ (K varies over all the compact subsets of G) is a neighborhood base of $\mathbb{1}$ for the topology of $\widehat{G}$.
3. If G is discrete, then $\widehat{G}$ is compact.
4. When G is a locally compact abelian group, the group $\widehat{G}$ is locally compact.

Proof. (1) ($\implies$) Clear. ($\impliedby$) Let $U \subset G$ open neighbourhood of e such that $\chi(U) \subset N(1)$. For every $m \in \mathbb{Z}_{\geq 1}$, by continuity of the product operation of G, there exists V open neighbourhood of e such that $V^{(m)} \subset U$. Then $\chi(V^{(m)}) \subset N(1/m)$ by the above lemma.

(2) We need to show that for every $K_1 \subset G$ compact and for every positive m, there exists $K \subset G$ a compact subset such that $W(K, N(1)) \subseteq W(K_1, N(1/m))$. WLOG, we can assume that $e \in K_1$ since $K_1 \cup \{e\}$ is again compact and $W(K_1, N(1/m)) = W(K_1 \cup \{e\}, N(1/m))$.

Let $K = K_1^{(m)}$, which is compact. It is clear that $W(K, N(1)) \subset W(K_1, N(1/m))$.

(3) When G is discrete then $\widehat{G} = \mathrm{Hom}(G, S^1)$ is the set of all group homomorphisms $G \longrightarrow S^1$. Morover, the compact-open topology of $C(G, S^1) = (S^1)^G$ conicides with the product topology on $(S^1)^G$. By Tychonoff's theorem $C(G, S^1)$ is compact and hence $\widehat{G}$ is compact as it is closed in $C(G, S^1)$.

(4) By (2) it suffices to show that for any $K \subset G$ neighbourhood of e, $W = W(K, \overline{N(1/4)})$ is a compact neighbourhood of $\mathbb{1}$ in $\widehat{G}$. Let $G_0 = G$ as groups with the discrete topology. Define $W_0 = \{\chi \in \widehat{G_0} : \chi(K) \subset \overline{N(1/4)}\}$. By part (1), $W_0 \subset W$ and certainly $W \subset W_0$. Hence $W = W_0$ (as sets). Now it is sufficient to prove that the induced topology τ_0 on W_0 by $\widehat{G_0}$ is finer than the induced topology τ on W by $\widehat{G}$ (since W_0 is clearly compact w.r.t τ_0).

Let $K_1 \subset G$ be compact. And consider $W' = W(K_1, N(1)) \cap W$. By (2), it is sufficient to prove that this is open in relative topology τ_0 or equivalently there exists an open τ_0-neighbourhood around e contained in W'. Let V be a neighbourhood of e such that $V^{(2)} \subset K$.

Since K_1 is compact, there exists a finite set such that $F \cdot V \supset K_1$. Let $W_0' = W_0(F, N(1/2)) \cap W$. We check that $W_0' \subset W'$. If $\mu \in W_0'$ then $\mu \in W$ such that $\mu(F) \subset N(1/2)$. Now $\mu(K_1) \subset \mu(F \cdot V) \subset N(1/2)N(1/2) = N(1)$. So $\mu \in W'$. □

In proof of the local compactness of $\widehat{G}$, we compared two topologies on it. For a completely different proof using the Arzela-Ascoli theorem, see [Keia]. Now we begin our proof of Pontryagin duality. Recall that we have a natural map

$$\mathrm{ev}_G : G \longrightarrow \widehat{\widehat{G}}, \qquad g \longmapsto (\mathrm{ev}_G(g) : \chi \longmapsto \chi(g))$$

Pontryagin duality states that ev_G induces an isomorphism of topological groups. It is easy to check that ev_G is a homomorphism of groups. So we only need to check bijectivity and topological properties. First we prove that ev_G is injective continuous open map. But before moving onto proof of this, we note that locally compact spaces in general are not normal (deleted Tychonoff plank is a standard counterexample) but still they satisfy a weaker version of the Uryshon's lemma:

Theorem 4.3. [Uryshon's lemma] ([Rud70], theorem 2.12) Suppose X is a locally compact Hausdorff space, $V \subset X$ open, and $K \subset X$ compact. Then there exists a function $f \in C_c(X)$ such that $f|_K = 1$ and $\mathrm{supp}(f) \subset U$.

Lemma 4.4. The mapping ev_G defined above is injective: that is, G separates points in $\widehat{G}$.

Proof. Suppose that $g \neq e$. It suffices to show the existence of a character χ such that $\chi(g) \neq 1$. Suppose that no such χ exists. Then by the left-invariance of the Haar measure we have

$$\widehat{f} = (L_g f)\widehat{\ } \qquad \text{for all } f \in L^1(G).$$

Hence by the Fourier inversion formula (3.4) we get $f = L_g f$ for all f in $V^1(G)$. Now, since G is Hausdorff, there exists an open neighborhood U of the identity such that $U \cap (g^{-1}U) = \varnothing$. By Uryshon's lemma, there exists a function $f \in C_c(G)$ such that $f(e) = 1$ and $\mathrm{supp}(f) \subset U$. Now we see that $f' = f * \widetilde{f}$ is a continuous function of positive type: For all $h \in C_c(G)$

$$\begin{aligned}\iint (f * \widetilde{f})\,(s^{-1}t)h(s)ds\,\overline{h(t)}dt &= \iint \langle L_{s^{-1}t} f, f\rangle\,(s^{-1}t)h(s)ds\,\overline{h(t)}dt \\ &= \iint \langle h(s)L_s f, h(t)L_t f\rangle\; ds\; dt \geq 0\end{aligned}$$

Also $\mathrm{supp}(f') \subset U$. But for such f', it is impossible that $f' = L_g f'$. Contradiction! □

Now let $\widehat{K}$ be a compact neighborhood of $\mathbb{1}$ and $V \subset S^1$ open, we define:

$$W(\widehat{K}, V) = \{\psi \in \widehat{\widehat{G}} : \psi(\chi) \in V \text{ for all } \chi \in \widehat{K}\} \quad \text{and} \quad W_G(\widehat{K}, V) = W(\widehat{K}, V) \cap \mathrm{ev}_G(G)$$

By lemma 4.4 we can regard $W_G(\widehat{K}, V)$ as a subset of G. Now we see the following:

Proposition 4.5. $W_G(\widehat{K}, V)$ and its translates constitute a base for the topology of G.

Proof. Let U be an open neighborhood of e. By Uryshon's lemma there exists a continuous positive type function g on G of with $\mathrm{supp}(g) \subset U$ and $g(e) = 1$. Now $\widehat{g}$ is positive (Fourier transform of a positive function is again positive). Hence by the inversion formula, we have

$$\int \hat{g}(\chi) d\chi = 1$$

Thus we may identify $\widehat{g}(\chi)d\chi$ with a finite Radon measure on $\widehat{G}$, which in particular, is inner regular. So given any $\epsilon > 0$, there exists a compact subset $\widehat{K}$ of $\widehat{G}$ such that

$$\int_{\widehat{K}} \widehat{g}(\chi) d\chi > 1 - \epsilon$$

and hence the corresponding integral over $\widehat{K}^c$ is less than ϵ. Now consider

$$g(y) = \int_{\widehat{K}} \widehat{g}(\chi)\chi(y) d\chi + \int_{\widehat{K}^c} \widehat{g}(\chi)\chi(y) d\chi$$

given by the Fourier inversion formula. As V shrinks to a sufficiently small neighborhood of 1 in S^1, the first integral above eventually lies within ϵ of unity for all $y \in W_G(\widehat{K}, V)$, while the second is unconditionally bounded in absolute value by ϵ. Hence $g > 1 - 2\epsilon$ on $W_G(\widehat{K}, V)$. But $\mathrm{supp}(g) \subset U$, and therefore $W_G(\widehat{K}, V) \subset U$. □

Corollary 4.6. The mapping ev_G is a homeomorphism onto its image.

Proof. By construction we have $\mathrm{ev}_G(W_G(K, V)) = W(K, V) \cap \mathrm{ev}_G(G)$. The above proposition (4.5) shows that ev_G is bicontinuous at e. Since ev_G is clearly a group isomorphism onto its image, so ev_G is continuous at every point of G by translation. □

Corollary 4.7. The image G a is closed in $\widehat{\widehat{G}}$.

Proof. **First we see that a locally compact and dense subset D of a Hausdorff space X must be open:** Since D is locally compact, there is a compact $K \subset D$ such that there is an open set U of X containing p such that $U \cap D \subset K$. Now since D is dense in X, if the open set $U - K$ was nonempty it would contain a member of D, which contradicts $U \cap D \subset K$. So $U \subset K \subset D$ and D contains a neighbourhood of p.

Now $\mathrm{ev}_G(G)$ is locally compact subgroup and is dense in $\overline{\mathrm{ev}_G(G)}$. So $\mathrm{ev}_G(G)$ is open subgroup hence also closed subgroup of $\overline{\mathrm{ev}_G(G)}$. So $\mathrm{ev}_G(G)$ is closed in $\widehat{\widehat{G}}$. □

Now we are reduced to showing that $\mathrm{ev}_G(G)$ is dense in $\widehat{\widehat{G}}$. This will require some further delicate analysis.

Let $f \in L^1(G) \cap L^2(G)$. As ususal define $\widetilde{f}(x) := f(x^{-1})$ for $x \in G$. An easy calculation shows that $\widehat{\widetilde{f}}(\chi) = \overline{\widehat{f}(\chi)}$. Now let $g = f * \widetilde{f}$. Then $g \in L^1(G)$ and is of positive type. Also easy to see that $\widehat{g} = \widehat{f} \cdot \widehat{\widetilde{f}}$. So by Fourier inversion (3.4), we get

$$\int |f(x)|^2 dx = g(1) = \int \widehat{g}(\chi) d\chi = \int |\widehat{f}(\chi)|^2 d\chi$$

This shows that the Fourier transform induces a map

$$L^1(G) \cap L^2(G) \longrightarrow L^2(\widehat{G}), \qquad f \longmapsto \widehat{f}$$

is an isometry onto its image. Let $\widehat{A_1}$ denotes the image. The following result is the key to our current discussion.

Lemma 4.8. $\widehat{A_1}$ is a dense subspace of the Hilbert space $L^2(\widehat{G})$.

Proof. Assume that $g \in L^2(\widehat{G})$ is orthogonal to every element in $\widehat{A_1}$. We will show that $g = 0$ in $L^2(\widehat{G})$. It is easy to see that $\widehat{A_1}$ is stable under multiplication by elements of $\alpha(G)$: $\widehat{\alpha(y) \cdot f} = \widehat{L_y f}$. Hence for all $f \in \widehat{A}_1$ and $y \in G$, we have that

$$\int g(\chi)\overline{f}(\chi)\chi(y) d\chi = 0.$$

This says that the Fourier transform of the measure $g(\chi)\overline{f}(\chi)d\chi$ is zero, and hence $g\overline{f}$ almost everywhere (proposition 3.6). But for a character χ we have $\widehat{(\chi \cdot f)} = L_\chi \widehat{f}$. Thus given any nonzero continuous element of $\widehat{A}_1$, we can produce an element of $\widehat{A}_1$ that does not vanish in some neighborhood of χ. Hence if the product $g\overline{f}$ is zero almost everywhere, it must be that g is zero almost everywhere which means g is zero in $L^2(\widehat{G})$. □

Also $L^1(G) \cap L^2(G)$ is dense in $L^2(G)$ (for $1 \leq p < \infty$, $C_c(G)$ is dense in $L^p(G)$, see [Rud70], theorem 3.14). So the isometry defined above may be extended by continuity to an isometry

$$L^2(G) \longrightarrow L^2(\widehat{G}) \qquad f \longmapsto \widehat{f}.$$

This extended Fourier transform called the *Plancherel transform*. Thus we have established:

Theorem 4.9. [Plancherel] Let G be a locally compact abelian group. Then the Plancherel transform defines an isometric *isomorphism* of Hilbert spaces from $L^2(G)$ onto $L^2(\widehat{G})$.

This theorem gives us following two easy corollries whose proof we leave to the reader.

Corollary 4.10. [Parseval's Identity] For all $f, g \in L^2(G)$, we have

$$\int f(x)\overline{g}(x) dx = \int \widehat{f}(\chi)\overline{\widehat{g}}(\chi) d\chi.$$

Corollary 4.11. Let $f, g \in L^2(G)$, and let $h \in L^1(G)$. Then if $h = f \cdot g$, we have $\widehat{h} = \widehat{f} * \widehat{g}$.

Now we come to our main proposition:

Proposition 4.12. Let U be a nonempty open subset of $\widehat{G}$. Then there exists a nonzero function $f \in L^1(G)$ such that $\widehat{f} \in L^1(\widehat{G})$ is a function with support contained in U.

Proof. Let μ be the Haar measure on $\widehat{G}$. By inner regularity, there exists a compact set K_1 with $\mu(K) > 0$. Then $K \subset \cup_{g \in G} sU$. By left invariance, we get that $\mu(U) > 0$. Thus, again by inner regularity, there exists a compact subset K of U with positive measure.

For every $x \in K$ we can find an open neighborhood V_x of e and an open neighborhood U_x of x such that $U_x V_x \subset U$. By compactness of K there are finitely many points $x_1, \ldots, x_n$ such that $K \subset \cup_i U_{x_i}$. Let $V = \cap_i V_{x_i}$ then $KV \subset U$. Consider the convolution $\mathbb{1}_K * \mathbb{1}_V$ of characteristic functions. From Plancherel theorem (4.9), there are functions $g_1, g_2 \in L^2(G)$ such that $\widehat{g_1} = \mathbb{1}_K$ and $\widehat{g_2} = \mathbb{1}_V$. Then $f = g_1 \cdot g_2 \in L^1(G)$ and by corollary 4.11, $\widehat{f} = \mathbb{1}_K * \mathbb{1}_V$. Moreover, it is easy to see that the integral of $\widehat{f}$ over $\widehat{G}$ is simply the product of the measures of $\mu(K) \cdot \mu(V)$, and hence positive. Thus $\widehat{f}$ is nonzero on a set of positive measure. □

Proof. **[Pontryagin 's Theorem]** As we observed above, it remains only to show that $\mathrm{ev}_G(G)$ is dense in $\widehat{\widehat{G}}$. If not, then by last proposition, there exists a function $\varphi \in L^1(\widehat{G})$ such that $\widehat{\varphi}$ is nonzero but $\widehat{\varphi}$ vanishes on $\mathrm{ev}_G(G)$. Let $\widehat{\chi}_0 \in \widehat{\widehat{G}}$. Then by definition,

$$\widehat{\varphi}(\widehat{\chi}_0) = \int \varphi(\chi)\widehat{\chi}_0(\chi^{-1})d\chi.$$

But the assumption that $\widehat{\varphi}$ vanishes on $\mathrm{ev}_G(G)$ means precisely that

$$\int \varphi(\chi)\chi(y^{-1})d\chi = 0$$

for all $y \in G$. Hence, as in the proof of Plancherel's theorem, $\varphi = 0$ almost everywhere (3.6), and therefore $\widehat{\varphi} = 0$. Contradiction! □

References

[Con19] John B Conway. *A course in functional analysis*. Vol. 96. Springer, 2019.

[Keia] *Local compactness of Character groups by Ascoli*. URL: https://kconrad.math.uconn.edu/blurbs/gradnumthy/loccptascoli.pdf.

[RV13] Dinakar Ramakrishnan and Robert J Valenza. *Fourier analysis on number fields*. Vol. 186. Springer Science & Business Media, 2013.

[Rud70] Walter Rudin. *Real and Complex Analysis P. 2*. McGraw-Hill, 1970.

[Keib] *The character group of* $\mathbb{Q}$. URL: https://kconrad.math.uconn.edu/blurbs/gradnumthy/characterQ.pdf.